1 Global existence and uniqueness theorem

Let $f(x, y)$ be

- Continuous as a function of two variables
- Lipschitz continuous in y: $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$

in the region $x \in [a, b]$, $y \in (-\infty, \infty)$. If $a \leq x_0 \leq b$, then the ODE

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1}$$

has exactly one solution $y = y(x)$ on the interval $a \leq x \leq b$.

2 Proof outline

1. The original ODE (1) is equivalent to

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$

2. Define the sequence $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$, where $y_0(x) = y_0$.
3. The sequence $y_n(x)$ converges to a function $y(x)$.
4. The function $y(x)$ is continuous.
5. The function $y(x)$ satisfies (2).
6. If another function $\tilde{y}(x)$ satisfies (2), then $\tilde{y}(x) = y(x)$.

3 Prerequisites

1. If $f(x)$ is continuous on the closed and bounded interval $[a, b]$, then it takes a maximum and minimum value on this interval and is bounded.
2. Compositions of continuous functions are continuous.
3. Integrals are continuous.
4. Differentiable functions are continuous.
5. Continuous functions are integral provided the integrals converge.
6. Bounded integral functions have finite integrals on bounded intervals.
7. First fundamental theorem of calculus
   (a) If $f(x)$ is continuous on $[a, b]$
   (b) Let $F(x) = \int_a^x f(x)\, dx$
   (c) Then $F(x)$ is differentiable on (a, b)
   (d) and $F'(x) = f(x)$ for all $x \in (a, b)$
8. Second fundamental theorem of calculus
   (a) If $F'(x) = f(x)$ for all $x \in (a, b)$
   (b) and $\int_a^b f(x)\, dx$ exists
   (c) then $\int_a^b f(x)\, dx = F(b) - F(a)$.

4 Part 1

4.1 (1) implies (2)

The proof is

$$y'(t) = f(t, y(t)) \tag{3}$$

$$\int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt \tag{4}$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\,dt \tag{5}$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{6}$$

Now I just need to show that the steps are valid.

1. It is valid to integrate to get (4).
   (a) $y(x)$ is continuous on $[a, b]$ since it is differentiable.
   (b) $y(x)$ is bounded on $[a, b]$ (continuous on bounded interval)
   (c) $f(x, y)$ is continuous on $[a, b] \times (-\infty, \infty)$ by assumption.
   (d) $f(x, y(x))$ is continuous (composition of continuous functions)
   (e) $f(x, y(x))$ is bounded on $[a, b]$ (continuous on bounded interval)
   (f) $f(x, y(x))$ can be integrated on any subset of $[a, b]$.
2. I can apply the second fundamental theorem of calculus to get (5).
   (a) $y'(x)$ is continuous on $[a, b]$ since $y'(x) = f(x, y(x))$ is continuous.

4.2 (2) implies (1)

The proof is

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{7}$$

$$\frac{d}{dx} y(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} f(t, y(t))\,dt\right) \tag{8}$$

$$y'(x) = f(x, y(x)) \tag{9}$$

Now I just need to show that the steps are valid.

1. (7) is differentiable.
   (a) $y(x)$ is continuous on $[a, b]$, since integrals are continuous.
   (b) $y(x)$ is bounded on $[a, b]$ (continuous on bounded interval)
   (c) $f(x, y)$ is continuous on $[a, b] \times (-\infty, \infty)$ by assumption.
   (d) $f(x, y(x))$ is continuous (composition of continuous functions)
   (e) $f(x, y(x))$ is bounded on $[a, b]$ (continuous on bounded interval)
   (f) $f(x, y(x))$ can be integrated on any subset of $[a, b]$.
   (g) The first fundamental theorem of calculus establishes differentiability.
2. The first fundamental theorem of calculus establishes (9).
3. The initial condition from (1) follows from letting $x = x_0$ in (7).

5 Part 2: Picard iteration

Define the sequence

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \qquad y_0(x) = y_0. \tag{10}$$

This is called Picard iteration. The idea behind the proof is that the Picard iterates have nice properties and converge to the solution of the ODE.

$$\begin{aligned}
y' &= y \qquad y(0) = 1 \\
f(x, y) &= y \\
y_0(x) &= 1 \\
y_1(x) &= 1 + \int_0^x y_0(t)\, dt = 1 + \int_0^x 1\, dt = 1 + x \\
y_2(x) &= 1 + \int_0^x y_1(t)\, dt = 1 + \int_0^x 1 + t\, dt = 1 + x + \frac{1}{2}x^2 \\
y_3(x) &= 1 + \int_0^x y_2(t)\, dt = 1 + \int_0^x 1 + t + \frac{1}{2}t^2\, dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
y(x) &= e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \cdots
\end{aligned}$$

6 Part 3

I need to show that $y_n(x) \to y(x)$ as $n \to \infty$. My tool for doing this is constructing bounds on $|y_{n-1}(x) - y_n(x)|$. If consecutive approximations get closer together, they must converge to something. This logic will be made rigorous later.

6.1 Bounding consecutive approximations

Note that $|y_0(x) - y_1(x)|$ is continuous on the closed and bounded interval $[a, b]$, so it must take on a maximum value in this interval. Thus, for some M, I can write $|y_0(x) - y_1(x)| \leq M$. The key to bounding subsequent intervals is to use Lipschitz continuity to relate consecutive differences using (assuming $x > x_0$)

$$\begin{aligned}
|y_n(x) - y_{n-1}(x)| &= \left| \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt - \int_{x_0}^{x} f(t, y_{n-2}(t))\, dt \right| && (11) \\
&= \left| \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))\, dt \right| && (12) \\
&\leq \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))|\, dt && (13) \\
&\leq \int_{x_0}^{x} K |y_{n-1}(t) - y_{n-2}(t)|\, dt && (14) \\
&= K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\, dt && (15)
\end{aligned}$$

This leads us to the following sequence of differences

$$
\begin{aligned}
|y_0(x) - y_1(x)| &\le M \\
|y_2(x) - y_1(x)| &\le K \int_{x_0}^{x} |y_1(t) - y_0(t)|\,dt = KM(x - x_0) \\
|y_3(x) - y_2(x)| &\le K \int_{x_0}^{x} |y_2(t) - y_1(t)|\,dt = K^2 M \int_{x_0}^{x} (x - x_0)\,dt = \frac{1}{2} K^2 M (x - x_0)^2 \\
|y_4(x) - y_3(x)| &\le K \int_{x_0}^{x} |y_3(t) - y_2(t)|\,dt = \frac{1}{2} K^3 M \int_{x_0}^{x} (x - x_0)^2\,dt = \frac{1}{3!} K^3 M (x - x_0)^3 \\
&\vdots \\
|y_n(x) - y_{n-1}(x)| &\le K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\,dt = \frac{1}{(n-2)!} K^n M \int_{x_0}^{x} (x - x_0)^{n-2}\,dt = \frac{1}{(n-1)!} K^{n-1} M (x - x_0)^{n-1}
\end{aligned}
$$

This bound was obtained assuming $x > x_0$, since otherwise many of the integrals are negative. I could repeat the argument with $x < x_0$. This gives me

$$|y_n(x) - y_{n-1}(x)| \le \frac{1}{(n-1)!} K^{n-1} M |x - x_0|^{n-1} \le \frac{1}{(n-1)!} K^{n-1} M (b - a)^{n-1}$$

6.2 Convergence

I can express the terms in the sequence as a telescoping sum

$$y_n(x) = y_0 + \sum_{k=1}^{n} (y_k(x) - y_{k-1}(x)). \tag{16}$$

If $y_n(x) \to y(x)$, then

$$y(x) = y_0 + \sum_{k=1}^{\infty} (y_k(x) - y_{k-1}(x)). \tag{17}$$

Note that this is just the definition of an infinite series. The limit exists if and only if the series converges. Since

$$
\begin{aligned}
\sum_{k=1}^{\infty} |y_k(x) - y_{k-1}(x)| &\le \sum_{k=1}^{\infty} \frac{1}{(k-1)!} K^{k-1} M (b - a)^{k-1} && (18) \\
&= M e^{K(b-a)} && (19)
\end{aligned}
$$

converges for all x, we conclude that the series converges absolutely for all $x \in [a, b]$. Since the series converges, $y_n(x) \to y(x)$.

7 Part 4

This is left as a homework assignment.

8 Part 5

So far, I have proven that the iterates defined by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt, \qquad y_0(x) = y_0. \tag{20}$$

converge pointwise to a continuous function $y_n(x) \to y(x)$. (Actually, part of that will be proven in homework.) Now I need to show that the result satisfies

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{21}$$

To see this choose any $x \in [a, b]$, so that

$$\begin{aligned}
\left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right| &= \left| \left(y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right) - \left(y_n(x) - y_0 - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right) \right| \\
&= \left| (y(x) - y_n(x)) - \left(\int_{x_0}^{x} f(t, y(t))\, dt - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right) \right| \\
&\le |y(x) - y_n(x)| + \left| \int_{x_0}^{x} f(t, y(t))\, dt - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right| \\
&\le |y(x) - y_n(x)| + \int_{x_0}^{x} |f(t, y(t)) - f(t, y_{n-1}(t))|\; dt \\
&\le |y(x) - y_n(x)| + \int_{x_0}^{x} K|y(t) - y_{n-1}(t)|\, dt \\
&\le |y(x) - y_n(x)| + (b - a)KB_{n-1},
\end{aligned}$$

where $|y(x) - y_{n-1}(x)| \le B_{n-1}$ is a bound with $B_n \to 0$ and B_n not depending on x. You will be asked to compute such a bound as it occurs in Theorem A. Since the same ideas apply here, I will omit the process of choosing B_n (I show how to do it in the homework solutions generally). It suffices that it exists. The simplest way to finish the argument is using a limit,

$$\begin{aligned}
\left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right| &= \lim_{n\to\infty} \left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right| \\
&\le \lim_{n\to\infty} \left(|y(x) - y_n(x)| + (b - a)KB_n \right) \\
&= \lim_{n\to\infty} |y(x) - y_n(x)| + (b - a)K \lim_{n\to\infty} B_n \\
&= 0 + (b - a)K(0) = 0
\end{aligned}$$

9 Part 6

To show uniqueness, assume that a solution $\tilde{y}(x)$ satisfies (2). Then, I want to show that $y(x) = \tilde{y}(x)$. Since all solutions are equal, there must be only one solution. This logic closely follows the logic of bounding the original series.

Note that $\tilde{y}$ is continuous (since it is a constant plus an integral). Since $|\tilde{y}(x) - y_0(x)|$ is continuous on the closed and bounded interval $[a, b]$, so it must take on a maximum value in this interval. Thus, for some

A, I can write $|\tilde{y}(x) - y_0(x)| \le A$. following the same logic (assuming $x > x_0$),

$$|\tilde{y}(x) - y_n(x)| = \left| \int_{x_0}^{x} f(t, \tilde{y}(t))\, dt - \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right| \tag{22}$$

$$= \left| \int_{x_0}^{x} f(t, \tilde{y}(t)) - f(t, y_{n-1}(t))\, dt \right| \tag{23}$$

$$\le \int_{x_0}^{x} |f(t, \tilde{y}(t)) - f(t, y_{n-1}(t))|\ dt \tag{24}$$

$$\le \int_{x_0}^{x} K |\tilde{y}(t) - y_{n-1}(t)|\, dt \tag{25}$$

$$= K \int_{x_0}^{x} |\tilde{y}(t) - y_{n-1}(t)|\, dt \tag{26}$$

This leads us to the following sequence of differences

$$\begin{aligned}
&|\tilde{y}(x) - y_0(x)| \le A \\
&|\tilde{y}(x) - y_1(x)| \le K \int_{x_0}^{x} |\tilde{y}(t) - y_0(t)|\, dt = KA(x - x_0) \\
&|\tilde{y}(x) - y_2(x)| \le K \int_{x_0}^{x} |\tilde{y}(t) - y_1(t)|\, dt = K^2 A \int_{x_0}^{x} (x - x_0)\, dt = \frac{1}{2} K^2 A (x - x_0)^2 \\
&|\tilde{y}(x) - y_3(x)| \le K \int_{x_0}^{x} |\tilde{y}(t) - y_2(t)|\, dt = \frac{1}{2} K^3 A \int_{x_0}^{x} (x - x_0)^2\, dt = \frac{1}{3!} K^3 A (x - x_0)^3 \\
&\qquad\qquad\vdots \\
&|\tilde{y}(x) - y_n(x)| \le K \int_{x_0}^{x} |\tilde{y}(t) - y_{n-2}(t)|\, dt = \frac{1}{(n-1)!} K^n A \int_{x_0}^{x} (x - x_0)^{n-1}\, dt = \frac{1}{n!} K^n A (x - x_0)^n
\end{aligned}$$

As before, I could repeat the argument with $x < x_0$. This gives me

$$|\tilde{y}(x) - y_n(x)| \le \frac{1}{n!} K^n A |x - x_0|^n \le \frac{1}{n!} K^n A (b - a)^n$$

Finally,

$$y(x) = \lim_{n \to \infty} y_n(x) = \tilde{y}(x).$$

This completes the proof.