Existence of Laplace transform

The Laplace transform $L[f(x)]$ exists provided the integral

$$\int_0^\infty f(x)e^{-px}\,dx = \lim_{a\to\infty}\int_0^a f(x)e^{-px}\,dx$$

exists for sufficiently large p.

1 Preliminary

1.1 Absolute convergence

If the integral

$$\int_a^b |f(x)|\,dx$$

converges, then the integral

$$\int_a^b f(x)\,dx$$

converges absolutely. Note that it is okay for a, b to be $\pm\infty$.

1.2 Comparison test

If $|f(x)| \leq g(x)$ for all $a \leq x \leq b$ and the integral

$$\int_a^b g(x)\,dx$$

converges, then the integral

$$\int_a^b f(x)\,dx$$

also converges absolutely.

1.3 Triangle inequality

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

1.4 Exponential order

The function $f(x)$ is said to have *exponential order* if there exist constants M, c, and n such that

$$|f(x)| \leq Me^{cx}$$

for all $x \geq n$.

2 Criteria for convergence (I)

The Laplace transform $L[f(x)]$ exists if it has exponential order and

$$\int_0^b |f(x)|\, dx$$

exists for any $b > 0$. Since we only need to show convergence for sufficiently large p, assume $p > c$ and $p > 0$.

$$\begin{aligned}
\int_0^\infty |f(x)e^{-px}|\, dx &= \int_0^n |f(x)e^{-px}|\, dx + \int_n^\infty |f(x)e^{-px}|\, dx \\
&\leq \int_0^n |f(x)|\, dx + \int_n^\infty e^{-px}|f(x)|\, dx \qquad 0 < e^{-px} \leq 1 \\
&\leq \int_0^n |f(x)|\, dx + \int_n^\infty e^{-px} M e^{cx}\, dx \qquad \text{exponential order} \\
&= \int_0^n |f(x)|\, dx + M\left[\frac{e^{(c-p)x}}{c-p}\right]_n^\infty \qquad p > c \\
&= \int_0^n |f(x)|\, dx + M\frac{e^{(c-p)n}}{p-c}
\end{aligned}$$

The first integral exists by assumption, and the second term is finite for $p > c$, so the integral

$$\int_0^\infty f(x)e^{-px}\, dx$$

converges absolutely and the Laplace transform $L[f(x)]$ exists.

3 Criteria for convergence (II)

The Laplace transform $L[f(x)]$ exists if:

1. $f(x)$ has exponential order and
2. on every closed interval $[0, b]$
   (a) $f(x)$ is bounded,
   (b) $f(x)$ is piecewise continuous, and
   (c) $f(x)$ has at most a finite number of discontinuities

Requirements 2(a-c) imply that

$$\int_0^b |f(x)|\, dx$$

will always exist, so we automatically satisfy criterion (I).

4 $F(p) \to 0$ as $p \to \infty$

Assume $f(x)$ satisfies criterion (I) This implies $F(p) = L[f(x)]$ will exist if if $p \geq m$ for some m. I want to show that $|F(p)|$ can be made arbitrarily close to 0 for sufficiently large p. Choose an $\epsilon > 0$. Fix a p. We will discover how large p needs to be as we go; we only care about $p \to \infty$, so we may choose p to be as large as we need.

$$|F(p)| = \left| \int_0^\infty f(x)e^{-px}\, dx \right| \leq \int_0^\infty |f(x)e^{-px}|\, dx = G(p).$$

Note that as $p \to \infty$, $e^{-px} \to 0$ for $x > 0$, so that I should be able to make the integral arbitrarily small for large p. The only potential complication is near $x = 0$, so we will need to deal with that separately. The important point here is that the part near 0 does not contribute very much to the integral. Let

$$K_a(p) = \int_a^\infty |f(x)e^{-px}|\, dx$$

Then, $G(p) = \lim_{a \to 0^+} K_a(p)$. By the definition of a limit, there exists an $\delta > 0$ such that

$$|K_a(p) - F(p)| < \frac{\epsilon}{2} \qquad \text{for all } 0 < a \leq \delta.$$

Using this (with $a = \delta$),

$$\int_0^\delta |f(x)e^{-px}|\, dx = F(p) - K_\delta(p) < \frac{\epsilon}{2}.$$

This lets me bound part of the integral.

$$|F(p)| \leq G(p) = \int_0^\delta |f(x)e^{-px}|\, dx + \int_\delta^\infty |f(x)e^{-px}|\, dx < \frac{\epsilon}{2} + \int_\delta^\infty |f(x)e^{-px}|\, dx.$$

If I assume $p > m$, then

$$\begin{aligned}
|F(p)| &< \frac{\epsilon}{2} + \int_\delta^\infty |f(x)e^{-px}|\, dx \\
&= \frac{\epsilon}{2} + \int_\delta^\infty |f(x)|e^{-(p-n)x}e^{-nx}\, dx \\
&\leq \frac{\epsilon}{2} + \int_\delta^\infty |f(x)|e^{-(p-n)\delta}e^{-nx}\, dx \qquad \text{since } x \geq \delta \\
&= \frac{\epsilon}{2} + e^{-(p-n)\delta} \int_\delta^\infty |f(x)|e^{-nx}\, dx
\end{aligned}$$

Criterion (I) gives us that

$$A = \int_\delta^\infty |f(x)|e^{-nx}\, dx \leq \int_0^\infty |f(x)|e^{-nx}\, dx$$

exists. Choose $p \geq n + \frac{1}{\delta}\ln\left(\frac{2A}{\epsilon}\right)$, so that

$$\begin{aligned}
|F(p)| &< \frac{\epsilon}{2} + Ae^{-(p-n)\delta} \\
&\geq \frac{\epsilon}{2} + Ae^{-\ln\left(\frac{2A}{\epsilon}\right)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since I can make $|F(p)|$ arbitrarily close to 0 for large p, I have $F(p) \to 0$ as $p \to \infty$.