Math 135-2, Homework 9

Name: ______________________________ ID: ______________

Problem 43.9

Consider the regular Sturm-Liouville problem consisting of equation (3) with the boundary conditions (9). Prove that every eigenfunction is unique except for a constant factor. Hint: Let $y = u(x)$ and $y = v(x)$ be eigenfunctions corresponding to a single eigenvalue λ, and use their Wronskian to show that they are linearly dependent on $[a, b]$. *You may find ideas of section 15 to be helpful, and you may freely use things from that and earlier sections.*

Problem 66.1

Find the extremals for the integral (1) if the integrand is

(a) $\dfrac{\sqrt{1+(y')^2}}{y}$

(b) $y^2 - (y')^2$

Problem 67.5

A uniform flexible chain of given length hangs between two points. Find its shape if it hangs in such a way as to minimize its potential energy.

Problem 67.6

Solve the original isoperimetric problem (Example 2) by using polar coordinates. Hint: Choose the origin to be any point on the curve and the polar axis to be the tangent line at that point; then maximize

$$\frac{1}{2}\int_0^\pi r^2\, d\theta$$

with the side condition that

$$\int_0^\pi \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

must be constant. Hint: Trying to solve the ODE first and filling in constants later is very difficult. Dealing with the constants first results in a much simpler ODE.

Problem A

Let a surface be parameterized as $\vec{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, where u and v are the

independent variables. Let $v = \hat{v}(u)$ parameterize a curve along this surface, which is a geodesic. What differential equation does $\hat{v}$ satisfy? Tip: use vector notation with $\vec{x}_u$ and $\vec{x}_v$; it the algebra a lot less tedious.

Problem B

The concept of an inner product can be extended to complex numbers by altering the requirements and definitions slightly. All quantities below may contain complex numbers. There is also some inconsistency on whether the first or second entry gets the complex conjugate; I will consistently conjugate the first.

$$\begin{aligned}
\langle u, v\rangle &= \overline{\langle v, u\rangle} \\
\langle au + bw, v\rangle &= \overline{a}\langle u, v\rangle + \overline{b}\langle w, v\rangle \\
\langle u, av + bw\rangle &= a\langle u, v\rangle + b\langle u, w\rangle \\
\langle u, u\rangle &\geq 0 \\
\langle u, u\rangle &= 0 \qquad \Longleftrightarrow \qquad u = 0
\end{aligned}$$

For vectors, the standard definition of an inner product is

$$\langle \mathrm{u}, \mathrm{v}\rangle = \mathrm{u}^\dagger \mathrm{v}.$$

Here $\mathrm{u}^\dagger = \overline{\mathrm{u}}^T$ is conjugate transpose. That is, conjugate all of the entries, then apply the usual transpose. This operation is called by many names and denoted by many symbols; I will call it the Hermitian transpose. Since conjugating a real number has no effect, the Hermitian transpose decays to the transpose when only real numbers are involved. In the case of functions,

$$\langle f, g\rangle = \int_a^b \overline{f} g \, dx.$$

As in the vector case, a complex conjugate is inserted. The reason for doing this is to ensure positivity. Indeed,

$$\langle f, f\rangle = \int_a^b \overline{f} f \, dx = \int_a^b |f|^2 \, dx \geq 0.$$

An operator is Hermitian if $\langle Au, v\rangle = \langle u, Av\rangle$, thus generalizing the notion of symmetry to complex numbers in a natural way. For matrices, this means $\mathrm{A}^\dagger = \mathrm{A}$.

(a) Both inner product versions can be generalized.

$$\langle \mathrm{u}, \mathrm{v}\rangle_{\mathrm{Q}} = \mathrm{u}^\dagger \mathrm{Q} \mathrm{v} \qquad \langle f, g\rangle_q = \int_a^b q \overline{f} g \, dx$$

What properties must the matrix Q and function $q(x)$ satisfy for these inner products to satisfy the properties above?

(b) Our interest in complex numbers is in showing that the eigenvalues of

$$(p(x)y'(x))' + (q(x)\lambda + r(x))y(x) = 0, \qquad y(a) = y(b) = 0$$

are real numbers. Let M be the operator defined by

$$(My)(x) = \frac{(p(x)y'(x))' + r(x)y(x)}{q(x)}$$

Note that My does not merely denote multiplication; it involves multiplying, dividing, and differentiating functions. What is the discrete analogue of M? (You may use D to denote the derivative operator; denote diagonal matrices with the corresponding capital letter.) Assume $p > 0$ and $q > 0$.

(c) Show that M is Hermitian, in both the continuous and discrete settings.

(d) Let $My = \lambda y$, where y and λ may be complex. By analyzing the quantity $\langle My, y\rangle_q$ and using the q-Hermitian property of M, show that λ is real.

(e) Show that if y is an eigenfunction corresponding to some eigenvalue λ, then $z = \overline{y} + y$ is a real-valued eigenfunction corresponding to the same eigenvalue. Since eigenfunctions are unique (see Problem 43.9), we may always assume that the eigenfunctions will be real.

Problem C

Let $\vec{x}(t)$ be a vector-valued function, which we want to choose to minimize

$$J = \int_a^b L(t, \vec{x}(t), \vec{x}'(t))\, dt,$$

subject to $\vec{x}(a) = \vec{x}_a$ and $\vec{x}(b) = \vec{x}_b$.
(a) What differential equation must be solved to minimize this? (They are listed in the book.)

(b) Derive the equivalent of the Beltrami identity. Why is it not as useful as it was for the scalar case?

Problem D

Lets consider the problem of minimizing

$$J = \int_a^b L(t, x(t), x'(t), x''(t))\, dt,$$

subject to $x(a) = x_a$, $x(b) = x_b$, $x'(a) = v_a$, and $x'(b) = v_b$. You may assume that all functions may be differentiated as many times as needed. Derive an extended version of the Euler-Lagrange differential equation. In general, what will the order of the resulting differential equation be?

Problem E

The curvature κ of a curve $y(x)$ is

$$\kappa = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}}.$$

We want to investigate curves that attempt to minimize curvature in some meaningful sense. The most obvious candidate is

$$J = \int_a^b \kappa \, ds,$$

subject to boundary conditions: $y(a) = y_a$, $y(b) = y_b$, $y'(a) = z_a$, $y'(b) = z_b$. Assume that $y(x)$ has as many derivatives as desired.

(a) Why is ds used instead of dx?

(b) Show that J depends on the boundary conditions but not the curve.

(c) What happens if the Euler-Lagrange equation is used to minimize this value anyway?