Math 135-2, Homework 8

Name: ______________________________ ID: ______________

Problem 42.1

If $w = F(x,y) = G(r,\theta)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right] \\ &= \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.\end{aligned}$$

Hint:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta \qquad \text{and} \qquad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta).$$

Similarly, compute $\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$ and $\frac{\partial^2 w}{\partial \theta^2}$.

Problem A

The Poisson integral in the case of a general disc is (see problem 42.3)

$$w(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(\phi)\, d\phi.$$

If $f(\phi)$ is continuous, show that the maximum and minumum values of $w(r,\theta)$ are obtained on the boundary. (It is possible that these values may be obtained in the interior and at the boundary.) How does this statement change if the continuity requirement is dropped? Hint: the substitution $\tan(\theta/2) = u$ is useful for integrating rational functions in $\sin\theta$ and $\cos\theta$. This substitution implies $\cos\theta = \frac{1-u^2}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$, and $d\theta = \frac{2\,du}{1+u^2}$, thus turning the integrand into a rational function.

Problem 43.1

The differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is called *exact* if it can be written in the form $[P(x)y']' + [S(x)y]' = 0$ for some function $S(x)$. In this case the second equation can be integrated at once to give the first order linear equation $P(x)y' + S(x)y = c_1$, which can then be solved by the method of Section 10. By equating coefficients and eliminating $S(x)$, show that a necessary and sufficient condition for exactness is $P''(x) - Q'(x) + R(x) = 0$.

Problem 43.3

If the equation in Problem 1 is not exact, it can be made exact by multiplying by a suitable

integrating factor $\mu(x)$. Thus, $\mu(x)$ must satisfy the condition that the equation $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$ is expressible in the form $[\mu(x)P(x)y']' + [S(x)y]' = 0$ for some function $S(x)$. Show that $\mu(x)$ must be a solution of the *adjoint equation*

$$P(x)\mu'' + [2P'(x) - Q(x)]\mu' + [P''(x) - Q'(x) + R(x)]\mu = 0.$$

In general (but not always), the adjoint equation is just as difficult to solve as the original equation. Find the adjoint equation in each of the folliwing cases:
(a) Legendre's equation: $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$
(b) Bessel's equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$

Problem 43.5

Show that the adjoint of the adjoint of the equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is the original equation.

Problem 43.6

The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is called *self-adjoint* if its adjoint is the same equation (except for notation).
(a) Show that this equation is self-adjoint if and only if $P'(x) = Q(x)$. In this case the equation becomes

$$P(x)y'' + P'(x)y' + R(x)y = 0$$

or

$$[P(x)y]' + R(x)y = 0,$$

which is the standard form of a self-adjoint equation.
(b) Which of the equations in Problem 3 are self-adjoint? (Just do (a) and (b).)

Problem 43.7

Show that any equation $P(x)y'' + Q(x)y' + R(x)y = 0$ can be made self-adjoint by multipyling through by

$$\frac{1}{P}e^{\int Q/P\,dx}.$$

Problem 43.9

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