Math 135-2, Homework 7

Name: ____________________________ ID: ______________

Problem 40.1

Find the eigenvalues λ_m and Eigenfunctions $y_n(x)$ for the equation $y'' + \lambda y = 0$ in each of the following cases:
(a) $y(0) = 0$, $y(\pi/2) = 0$
(f) $y(a) = 0$, $y(b) = 0$

Problem 40.2

If $y = F(x)$ is an arbitrary function, then $y = F(x + at)$ represents a wave of fixed shape that moves to the left along the x-axis with velocity a (Fig. 49). Similarly, if $y = G(x)$ is another arbitrary function, then $y = G(x - at)$ is a wave moving to the right, and the most general one-dimensional wave with velocity a is

$$y(x,t) = F(x + at) + G(x - at) \tag{1}$$

(a) Show that (2) **satisfies the wave equation (8).**
(b) It is easy to see that the constant a in equation (8) has the dimensions of velocity. Also, it is intuitively clear that if a stretched string is disturbed, then waves will move in both directions away from the source of the disturbance. These considerations suggest introducing the new variables $\alpha = x + at$ and $\beta = x - at$. Show that with these independent variables, equation (8) becomes

$$\frac{\partial^2 y}{\partial\alpha\partial\beta} = 0, \tag{2}$$

and from this derive (2) **by integration. Formula** (2) **is called *d'Alembert's solution* of the wave equation. It was also obtained by Euler, independently of d'Alembert but slightly later.**

Problem 40.3

Consider an infinite string stretched taut on the x-axis from $-\infty$ to ∞. Let the string be drawn aside into a curve $y = f(x)$ and released, and assume that its subsequent motion is described by the wake equation (8).
(a) Use (2) **to show that the string's displacement is given by *d'Alembert's formula*,**

$$y(x,t) = \frac{1}{2}[f(x + at) + f(x - at)]. \tag{3}$$

Hint: remember the initial conditions (11) and (12).
(b) Assume further that the string remains motionless at the points $x = 0$ and $x = \pi$ (such

points are called *nodes*), so that $y(0,t) = y(\pi,t) = 0$, and use (3) to show that $f(x)$ is an odd function that is periodic with period 2π [that is, $f(-x) = -f(x)$ and $f(x+2\pi) = f(x)$].
(c) Show that since $f(x)$ is odd and periodic with period 2π, it necessarily vanishes at 0 and π.
(d) Show that Bernoulli's solution (17) can be written in the form of (3). Hint: $2\sin nx \cos nat = \sin[n(x+at)] + \sin[n(x-at)]$.

Problem 41.4

In the preceding problem, find $w(x,t)$ if the ends of the rod are kept at $0°C$, $w_0 = 0°C$, and the initial temperature distribution is $f(x)$.

Problem 41.7

The two-dimensional heat equation is

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial w}{\partial t}$$

Use the method of separation of variables to find a steady-state solution of this equation in the infinite strip of the xy-plane bounded by the lines $x = 0$ and $x = \pi$, and $y = 0$ if the following conditions are satisfied:

$$w(0,y) = 0 \qquad w(\pi,y) = 0 \qquad w(x,0) = f(x) \qquad \lim_{y\to\infty} w(x,y) = 0$$

Problem A

The three-dimensional heat equation is

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial t}.$$

This describes a block of material, in this case $[0,\pi] \times [0,\pi] \times [0,\pi]$. The initial conditions are $w(x,y,z,0) = f(x)g(y)h(z)$. Boundary conditions are $w(0,y,z,t) = w(\pi,y,z,t) = w(x,0,z,t) = w(x,\pi,z,t) = w(x,y,0,t) = w(x,y,\pi,t) = w_1$. Solve for $w(x,y,z,t)$.

Problem B

Let $\phi_1, \phi_2, \phi_3, \ldots$ and $\theta_1, \theta_2, \theta_3, \ldots$ be two normalized orthogonal sequences, in the sense of

$$\langle \phi_m, \phi_n \rangle_x = \int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\langle \theta_m, \theta_n \rangle_y = \int_c^d \theta_m(y)\theta_n(y)\,dy = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Let $\sigma_{i,j}(x,y) = \phi_i(x)\theta_j(y)$ be a sequence of functions in two dimensions (it does not matter the order in which they are enumerated).

(a) It is possible to construct an inner product $\langle \sigma_{i,j}, \sigma_{m,n} \rangle_{xy}$ under which this sequence is normalized and orthogonal. Construct this inner product and show that the sequence is normalized and orthogonal with respect to it.
(b) It is desired to express $f(x,y)$ as in terms of this basis. That is,

$$f(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \sigma_{i,j}(x,y).$$

How should $a_{i,j}$ be selected?
(c) Show how the solution from Problem A can be modified to handle the more general initial condition $w(x,y,z,0) = r(x,y,z)$. You may assume the sequence $\sigma_{i,j}$ is complete.