Math 135-2, Homework 6

Name: ______________________________ ID: ______________

Problem 37.1

One of the important consequences of the orthogonality properties of the trigonometric sequence (2) [namely, equations (4) in this section and (2), (5), (6), (8) in section 33] is *Bessel's inequality*: If $f(x)$ is any function integrable on $[-\pi, \pi]$, its ordinary Fourier coefficients satisfy the inequality

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx. \tag{1}$$

Prove this by the following steps:
(a) For any $n \geq 1$, define

$$s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

and show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)s_n\,dx = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

(b) By considering all possible products in the multiplication of $s_n(x)$ by itself, show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[s_n(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

(c) By writing

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x) - s_n(x)]^2\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx - \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)s_n(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}[s_n(x)]^2\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx - \frac{1}{2}a_0^2 - \sum_{k=1}^{n}(a_k^2 + b_k^2)\end{aligned}$$

conclude that

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx,$$

and from this complete the proof.

Observe that the convergence of the series on the left side of (1) implies the following corollary of Bessel's inequality: If a_n and b_n are the ordinary Fourier coefficients of $f(x)$, then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Problem 37.3

Prove the Schwarz inequality (19) $[|(f,g)| \le \|f\|\|g\|]$. *Hint:* If $\|g\| \ne 0$, then the function $F(\alpha) = \|f + \alpha g\|^2$ is a second degree polynomial in α that has no negative values; examine the discriminant.

Problem 34.6

Mathematicians prefer the classes of functions they study to be linear space, that is, to be closed under the operations of addition and multiplication by scalars. Unfortunately, this is not true for the class of functions defined on the interval $-\pi \le x < \pi$ that satisfy the Dirichlet conditions. Verify this statement by examining the functions

$$f(x) = x^2 \sin\frac{1}{x} + 2x \qquad (x \ne 0), \qquad f(0) = 0$$

and

$$g(x) = -2x.$$

Problem 38.1

Consider the sequence of functions $f_n(x)$, $n = 1, 2, 3, \ldots$, defined on the interval $[0,1]$ by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le 1/n, \\ \sqrt{n}, & 1/n < x < 2/n, \\ 0, & 2/n \le x \le 1. \end{cases}$$

(a) Show that the sequence $\{f_n(x)\}$ converges to the zero function on the interval $[0,1]$.
(b) Show that the sequence $\{f_n(x)\}$ does *not* converge in the mean to the zero function on the interval $[0,1]$.

Problem 38.2

Consider the following sequence of closed subintervals of $[0,1]$: $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$, $[0,\frac{1}{4}]$, $[\frac{1}{4},\frac{1}{2}]$, $[\frac{1}{2},\frac{3}{4}]$, $[\frac{1}{4},1]$, $[0,\frac{1}{8}]$, $[\frac{1}{8},\frac{1}{4}]$, $\ldots$, and denote the nth subinterval by I_n. Now define the sequence of functions $f_n(x)$ on $[0,1]$ by

$$f_n(x) = \begin{cases} 1 & \text{for } x \text{ in } I_n, \\ 0 & \text{for } x \text{ not in } I_n. \end{cases}$$

(a) Show that the sequence $\{f_n(x)\}$ converges in the mean to the zero function in the interval $[0,1]$.
(b) Show that the sequence $\{f_n(x)\}$ does *not* converge pointwise at any point in the interval $[0,1]$.

Problem 38.5

The function $f(x) = x$ to be approximated on $[0, \pi]$ by

$$p(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

in such a way that $\int_0^\pi [x - p(x)]^2 \, dx$ is minimized. What values should the coefficients of b_k have?

Problem A

Operations like differentiation and integration are called *operators*. They take one function and give you another one. For example, let D be the derivative operator, so that $Df = f'$. Then, $Lf = D(Df) = f''$ is the Laplacian operator, which in the case of real functions is just the second derivative. An operator A is called *symmetric* if $\langle Af, g \rangle = \langle f, Ag \rangle$ for all f and g, where $\langle u, v \rangle$ represents an inner product.
(a) In the case where f and g are ordinary vectors in $\mathbb{R}^n$ with the usual inner product $\langle \mathrm{u}, \mathrm{v} \rangle = \mathrm{u} \cdot \mathrm{v}$, operators A are just matrices. Show that in this case, the above definition of symmetric corresponds to the usual definition of a symmetric matrix.
(b) For what follows, we will use the inner product defined on $[a, b]$ by

$$\langle u, v \rangle = \int_a^b u(x)v(x) \, dx.$$

Show that the Laplacian operator is symmetric on $[a, b]$, provided we limit ourselves to the space of function f defined on $[a, b]$, where $f(a) = f(b) = 0$ and f has continuous second derivatives on $[a, b]$. (The definition of symmetric requires a property be satisfied for all f and g; only f and g with the above properties are to be tested.)

(c) A symmetric operator A is called *positive definite* if for all f we have $\langle Af, f \rangle \geq 0$, with $\langle Af, f \rangle = 0$ only when $f = 0$. Show that $-L$, defined by $(-L)f = -f''$, is positive definite, under the same restrictions on f.