Math 135-2, Homework 4

Name: ______________________________ ID: ______________

Problem 68.1

Find the exact solution of the initial value problem

$$y' = y^2, y(0) = 1.$$

Starting with $y_0(x) = 1$, apply Picard's method to calculate $y_1(x)$, $y_2(x)$, $y_3(x)$, and compare these results with the exact solution.

Problem 68.3

It is instructive to see how Picard's method works with a choice of the initial approximation other than the constant function $y_x(x) = y_0$. Apply the method to the initial value problem (4) $[y' = x + y,\ y(0) = 1]$ with
(a) $y_0(x) = e^x$

Problem A

The proof of Picard's theorem relies on the following logic.

1. **There is a sequence of functions $\{y_n\}$, where $y_n(x) = y_0(x) + \sum_{k=1}^{n} (y_k(x) - y_{k-1}(x))$.**
2. **For each x, $|y_k(x) - y_{k-1}(x)| \leq a(Kh)^{k-1}$.**
3. **The series $\sum_{k=1}^{n} a(Kh)^{k-1}$ converges, so $y_n(x) \to y(x)$ for each x.**
4. **Each y_n is continuous.**
5. **The bound on $|y_n(x) - y(x)|$ does not depend on x.**
6. **Therefore, y is also continuous.**

The logic here is important, but at the same time it is a bit subtle. Here, you are being asked to fill in some of the details. Remember that you are filling in details of a proof, so you need to be precise. You may find it helpful to refer back to theorems from calculus. You may assume as given any theorems from calculus, but you should not assume anything beyond that.
(a) Explain for step 3 why convergence of that series implies $y_n(x) \to y(x)$.
(b) Why is each y_n continuous in step 4?
(c) Give an explicit bound on $|y_n(x) - y(x)|$ for step 5; your bound should not depend on x.
(d) Prove that steps 4 and 5 imply that y is a continuous function.

Problem B

Consider the following two statements:

1. **$f(x)$ has continuous derivatives on interval I.**
2. **$f(x)$ is Lipschitz continuous on interval I.**

For each of the following intervals, determine whether (i) condition 1 implies condition 2 or (ii) condition 2 implies condition 1. In each case, if the implication fails, provide a counterexample. If the condition is true, explain briefly why it is true. Note that it may be the case that both or neither implication is true.
(a) $I = (a, b)$
(b) $I = [a, b]$
(c) $I = (-\infty, \infty)$

Problem C

Assume that $f(x)$ is Lipschitz continuous on $[a, b]$ and $[b, c]$. Show that it is also Lipschitz continuous on $[a, c]$. Note that $f(x)$ might not be differentiable.