

# Math Review

CS 130

## 1. Points

- Locations:  $P, Q, R$

## 2. Vectors

- Direction, magnitude
- No location
- Used to indicate quantities like change in position, velocity, force, etc.
- $\mathbf{u}, \mathbf{v}, \mathbf{w}$  or  $\vec{u}, \vec{v}, \vec{w}$
- Note that positions are very often represented with vectors (displacement from the origin).
- Vector operations

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \quad k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

- Coordinates;  $a_1, a_2, a_3$  are coordinates,  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are a basis

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

- In practice, the basis is normally understood (usually the standard one above), and only the coordinates are written explicitly.

## 3. Linear operator

- Special type of function  $f(x)$
- Defining property:  $af(x) + bf(y) = f(ax + by)$
- Typical form  $f(x) = ax$ , though not always. (Differentiation is a linear operator.)
- For example, if  $x, y$  are vectors, then the linear operators are matrices. That is,  $f(\mathbf{u}) = \mathbf{M}\mathbf{u}$ .
- Note that  $f(0) = 0$ .

## 4. Affine operator

- Typical form:  $f(x) = ax + b$ .
- Note that  $f(0)$  is not necessarily 0.
- The definition is fairly general.  $x$  and  $b$  could be vectors;  $a$  could be a matrix.

## 5. Lines

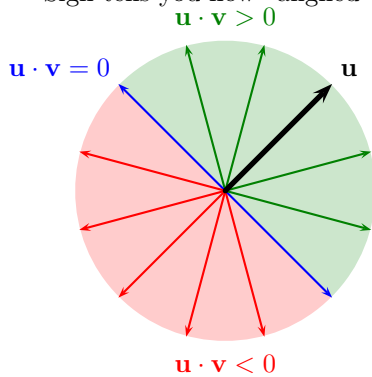
- $L(t) = P + t\mathbf{u}$ .
- Point on the line:  $P$
- Direction of the line:  $\mathbf{u}$
- $t$  is a parameter that tells where a point is along the line.

#### 6. Line segment

- Straight line connecting endpoints  $P, Q$ .
- $L(t) = (1 - t)P + tQ; t \in [0, 1]$ .
- Note:  $L(t) = (1 - t)P + tQ = P + t(Q - P)$ ,  $\mathbf{u} = Q - P$ .

#### 7. Dot product

- $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$
- Well defined for any size of vector (any number of components), but both vectors must have the same size.
- $\mathbf{u} \cdot \mathbf{u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1a_1 + a_2a_2 + a_3a_3 = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{u}\|^2$ ;  $\|\mathbf{u}\|$  is the length of the vector.
- $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta$ ;  $\theta$  is the angle between the vectors.
- Sign tells you how “aligned” two vectors are:



#### 8. Cross Product

- Defined in 3D only.

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= \underbrace{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}_{\text{(notation abuse)}} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

- Note that  $\mathbf{w} = \mathbf{u} \times \mathbf{v}$  is a vector.

- Length:  $\|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$ .
- Direction:  $\mathbf{w}$  is orthogonal to  $\mathbf{u}$  and  $\mathbf{v}$ :  $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$ .
- Note that both  $\mathbf{w}$  and  $-\mathbf{w}$  have these properties; need a convention to disambiguate direction.
- How I remember it:  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ .
- Others use “right hand rule” and similar devices.
- Note that  $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ , since  $\theta = 0$  and  $\|\mathbf{u} \times \mathbf{u}\| = \|\mathbf{u}\|\|\mathbf{u}\| \sin \theta = 0$

9. Dot product and cross product have lots of useful properties, which are straightforward to check:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} & (a\mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v}) & (\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} \\ \mathbf{u} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{u} & (a\mathbf{u}) \times \mathbf{v} &= \mathbf{u} \times (a\mathbf{v}) = a(\mathbf{u} \times \mathbf{v}) & (\mathbf{u} + \mathbf{w}) \times \mathbf{v} &= \mathbf{u} \times \mathbf{v} + \mathbf{w} \times \mathbf{v} \\ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} & \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 &= (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 \end{aligned}$$

10. Matrices

- Table of numbers

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}$$

- Entries are indexed  $a_{ij}$ ; row is  $i$ , column is  $j$ .
- Matrices have dimensions that indicate their size and shape:  $\mathbf{M}$  is  $3 \times 3$ ,  $\mathbf{N}$  is  $4 \times 2$ .
- Matrix-vector multiply:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 \end{pmatrix}$$

- Pattern:  $\mathbf{v} = \mathbf{M}\mathbf{u}$  is calculated as  $v_i = \sum_j m_{ij}u_j$
- Can be thought of as a linear operator:  $\mathbf{v} = f(\mathbf{u}) = \mathbf{M}\mathbf{u}$ .
- Composition:  $f(\mathbf{u}) = \mathbf{M}\mathbf{u}$ ,  $g(\mathbf{u}) = \mathbf{P}\mathbf{u}$ ,  $h(\mathbf{u}) = f(g(\mathbf{u}))$  or  $h = f \circ g$ .

$$\mathbf{w} = f(g(\mathbf{u})) = f(\mathbf{P}\mathbf{u}) = \mathbf{M}(\mathbf{P}\mathbf{u}) = (\mathbf{M}\mathbf{P})\mathbf{u} = \mathbf{Q}\mathbf{u}$$

$$w_i = \sum_j m_{ij} \left( \sum_k p_{jk} u_k \right) = \sum_k \underbrace{\left( \sum_j m_{ij} p_{jk} \right)}_{q_{ik}} u_k$$

$$\mathbf{Q} = \mathbf{M}\mathbf{P} \Leftrightarrow q_{ik} = \sum_j m_{ij} p_{jk}$$

- This is the rule for matrix multiplication.
- Transpose:  $\mathbf{N} = \mathbf{M}^T$  is defined by  $n_{ij} = m_{ji}$