

Linear Operators and 2D Transformations

Craig Schroeder

1 Linear Operators

A function f is a linear operator if it satisfies the property of superposition:

$$f(ax + by) = af(x) + bf(y)$$

In graphics, if x is a vector, this operation is represented as a matrix-vector multiplication:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \text{where} \quad y_i = \sum_k A_{ik}x_k$$

1.1 Composition of Transformations

Applying multiple transformations is equivalent to matrix multiplication. If we have $\mathbf{x} \xrightarrow{\mathbf{A}} \mathbf{y} \xrightarrow{\mathbf{B}} \mathbf{z}$, then:

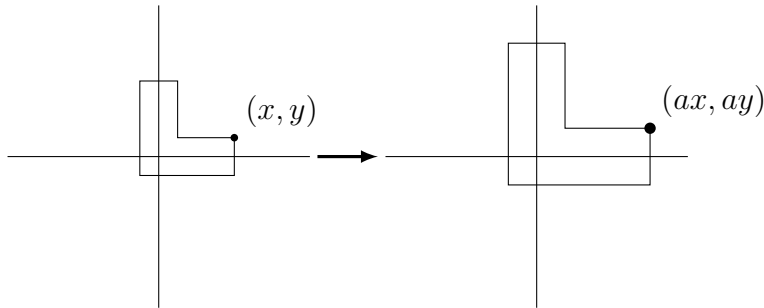
$$\mathbf{z} = \mathbf{B}(\mathbf{A}\mathbf{x}) = (\mathbf{B}\mathbf{A})\mathbf{x} = \mathbf{C}\mathbf{x}$$

The resulting matrix $\mathbf{C}$ is the product of $\mathbf{B}$ and $\mathbf{A}$. In fact, this can be understood as the *definition* of matrix-matrix multiplication.

2 2D Transformations

Common 2D transformations can be represented using 2×2 matrices.

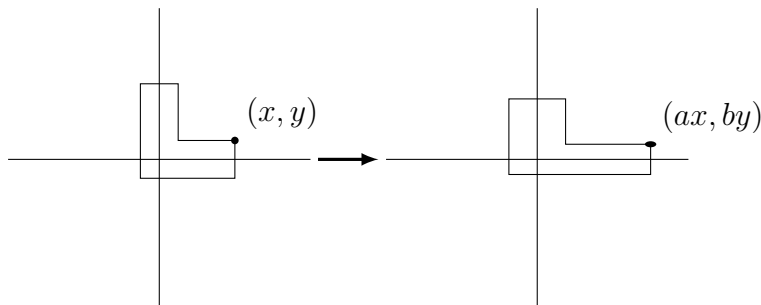
2.1 Uniform Scale



Scales both axes by the same factor a .

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix}$$

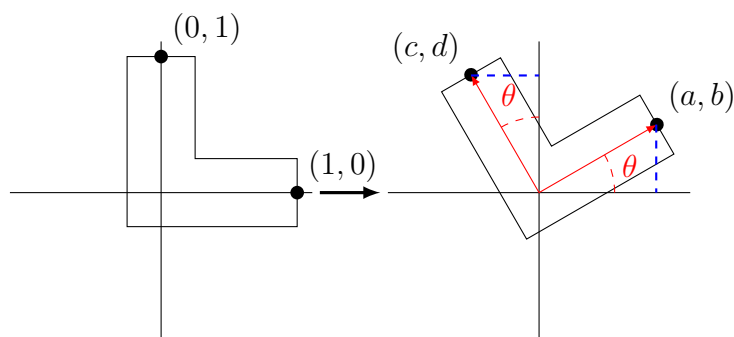
2.2 Nonuniform Scale



Scales x by a and y by b .

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

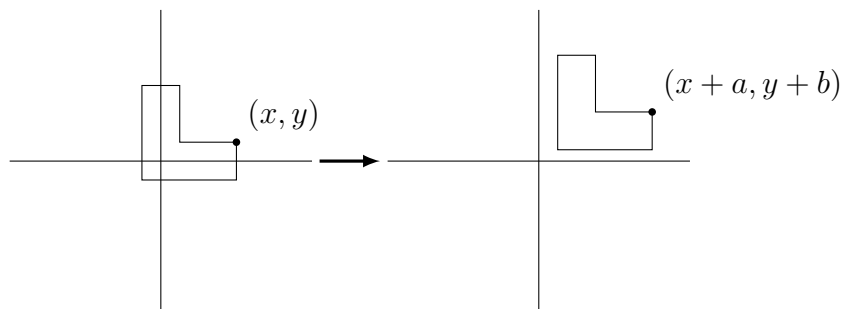
2.3 Rotation



The result of the transformation is less obvious in this case. We can observe the effects of the rotation by considering the points $(1, 0)$ and $(0, 1)$. (a, b) is on a triangle with angle θ , so that $a = \cos \theta$ and $b = \sin \theta$. Similarly for the point $(0, 1)$ we find $c = -\sin \theta$ and $d = \cos \theta$. We conclude the transformation matrix is

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

2.4 Translation



$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

This is a problem. No such matrix entries exist (the matrix cannot depend on x or y). This is especially obvious if we let $x = y = 0$. Multiplying a matrix by zero always gives zero, not (a, b) . We can use a little trick to get around this problem. We will always add a 1 to the end of our vectors.

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ 1 \end{pmatrix}$$

This is solvable.

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ 1 \end{pmatrix}$$

This is our transform matrix for translation. We can extend our 2×2 matrices in this way by adding a 1 to the bottom right.