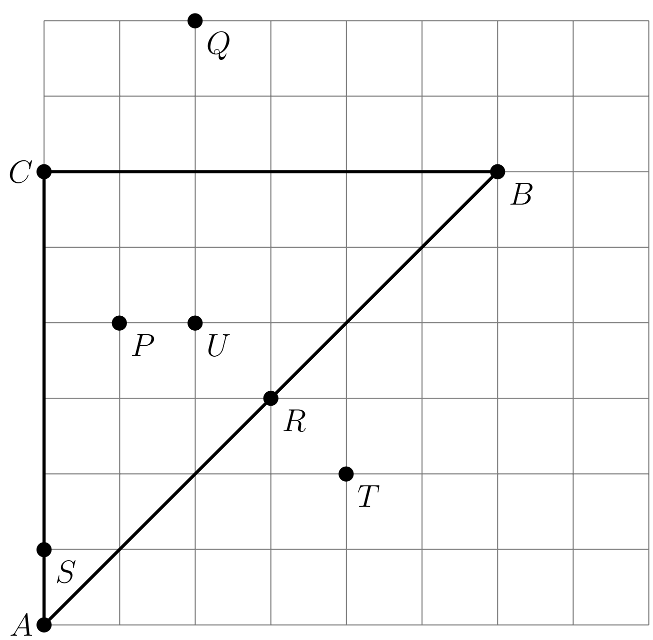
CS 130, Final

Solutions

Read the entire exam before beginning. **Manage your time carefully.** This exam has 59 points; you need 50 to get full credit. Additional points are extra credit. 59 points → 3.1 min/point. 50 points → 3.6 min/point.

Problem 1 (3 points)

For each set of barycentric weights below, label the corresponding point in the figure with those weights. You do not need to show your work, but your points must be plotted accurately to count.



Point	α	β	γ
A	1	0	0
B	0	1	0
C	0	0	1
P	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
Q	$-\frac{1}{3}$	$\frac{1}{3}$	1
R	$\frac{1}{2}$	$\frac{1}{2}$	0
S	$\frac{5}{6}$	0	$\frac{1}{6}$
T	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
U	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Problem 2 (2 points)

Which of these dominates the cost of ray tracing when acceleration structures are not in use and why? (a) Computing rays to cast, (b) calculating surface normals, (c) shading computations, (d) intersections, or (e) texture mapping.

If there are p pixels, n objects, and l lights, ignoring reflections or transparency, the operations above will execute approximately (a) $p + pl$, (b) p, (c) p, (d) $pn + pnl$, (e) p times. Of these, intersections are the most expensive.

Problem 3 (2 points)

Most color printers operate in CMYK space (C=cyan, M=magenta, Y=yellow, K=black) and have separate ink/toner cartridges for each of those four colors.
(a) Why are cyan, magenta, and yellow used instead of red, green, and blue?
(b) Why do these devices use a fourth color?

(a) RGB are used for additive color mixing, as occurs with lights. These colors primarily stimulate individual cones in the eye. CMY are used for subtractive color mixing, as occurs with pigments (and printing). These *absorb* individual colors.
(b) Black is used as a separate cartridge because most printed content is in black. Note that black could be obtained by mixing the other colors.

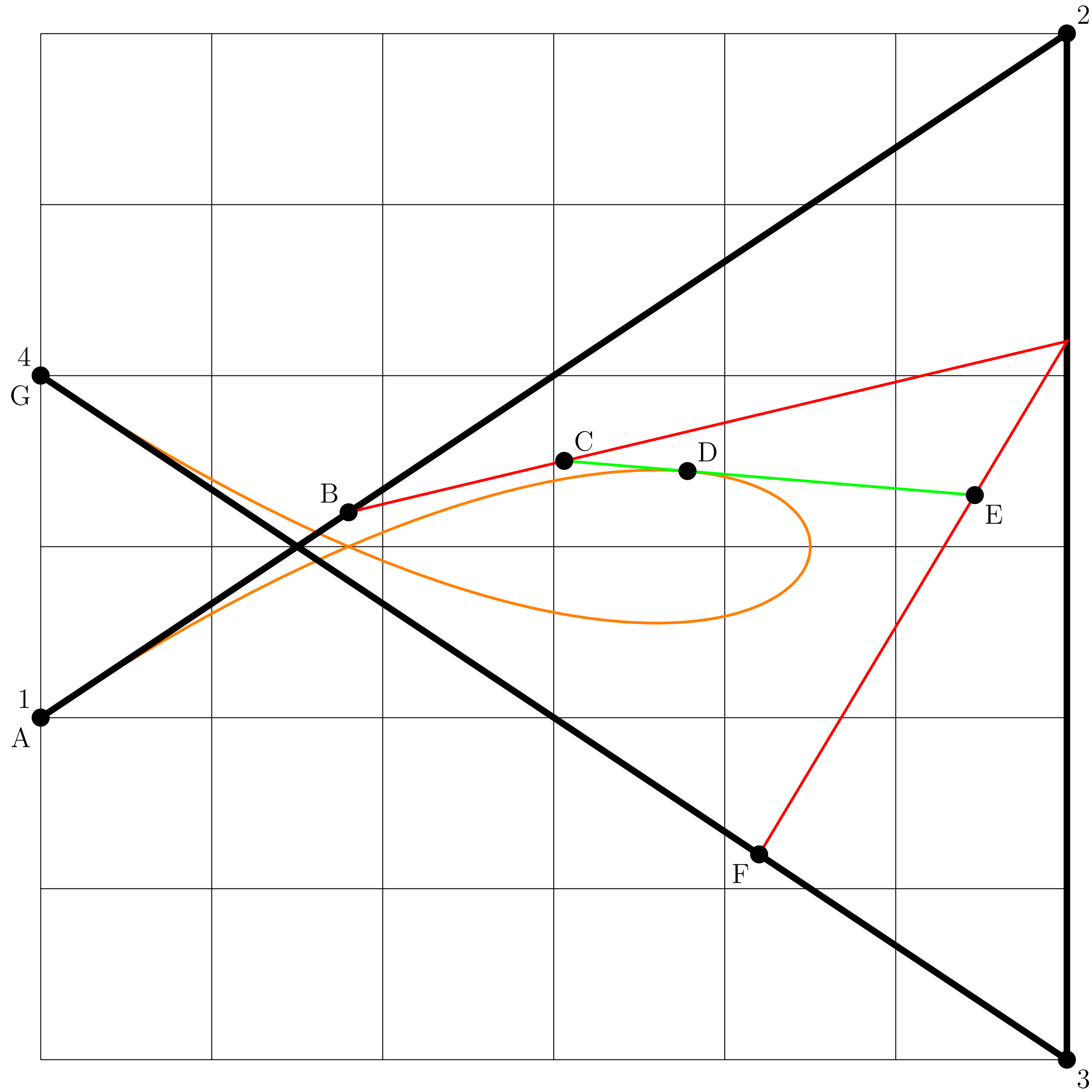
Problem 4 (2 points)

Plants absorb a portion of the sunlight that they receive and use it to produce sugars in a process called photosynthesis. What colors of light do you think the plants are using?

Plants appear green because they reflect the green light that they receive. They absorb the colors that are not green. They use red/orange light and blue/violet light for photosynthesis. They do not use green or yellow.

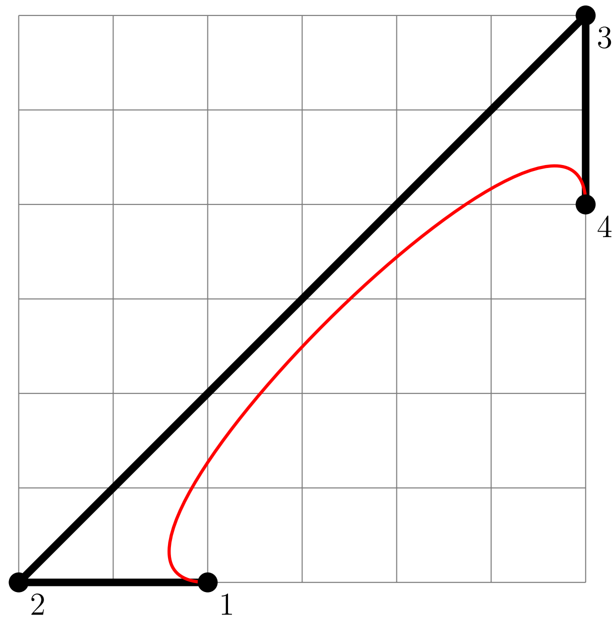
Problem 5 (3 points)

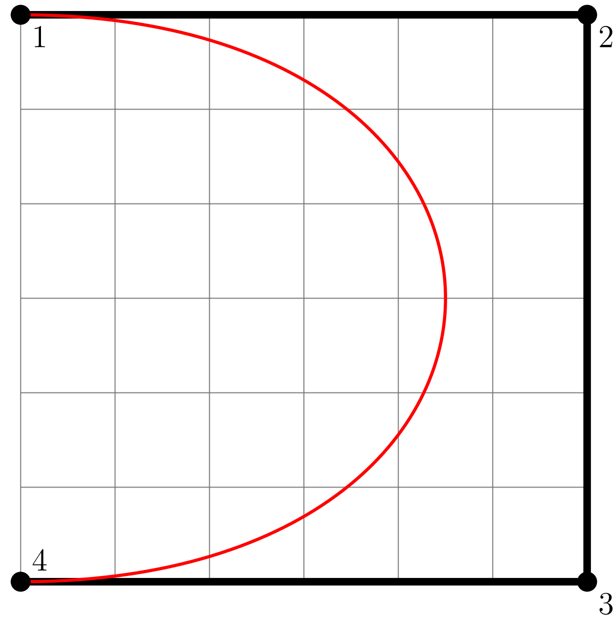
Geometrically subdivide the Bézier curve given by the control points below and at $t = 0.3$ along the curve. Label the new control points A, B, C, ..., G. (The two resulting Bézier curves should share their common point, so only 7 control points are required.)

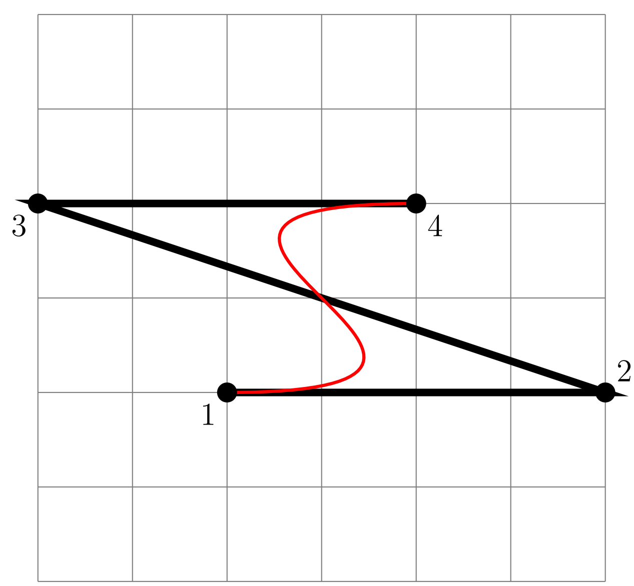

2
4
G
C
D
B
E
1
A
F
3

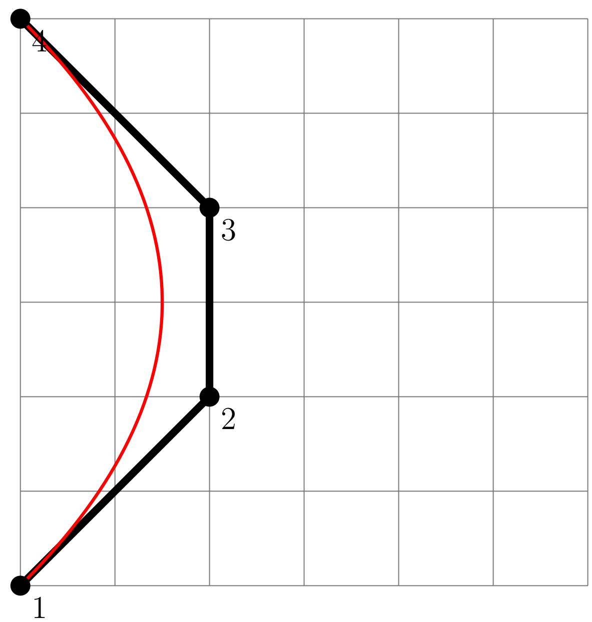
Problem 6 (2 points)

In each of the four examples below, control points are shown for a cubic Bezier curve. Sketch out approximately what these curves will look like.









Problem 7 (3 points)

Clip the segment (in homogeneous coordinates) with endpoints $A = (-1, 0, 1, 2)$ and $B = (-2, -1, 0, 1)$.

Note that both vertices satisfy all of the clipping constraints except $-w \leq x$, which B fails. Thus, we only need to clip against the plane $x = -w$. Let

$$\begin{aligned}
P &= (1-\gamma)A + \gamma B \\
0 &= P_x + P_w \\
&= ((1-\gamma)A_x + \gamma B_x) + ((1-\gamma)A_w + \gamma B_w) \\
&= ((1-\gamma)(-1) + \gamma(-2)) + ((1-\gamma)(2) + \gamma(1)) \\
&= -1 + \gamma - 2\gamma + 2 - 2\gamma + \gamma \\
&= 1 - 2\gamma \\
\gamma &= \frac{1}{2} \\
P &= \frac{1}{2}A + \frac{1}{2}B \\
P &= \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}
\end{aligned}$$

Then, AP is the clipped segment.

Problem 8 (2 points)

What are homogeneous coordinates and why does the graphics pipeline use them?

These are coordinates with an extra component (x, y, z, w) that are related to regular coordinates by $(x/w, y/w, z/w)$. The extra coordinate makes it possible to represent translations and projections as a matrix.

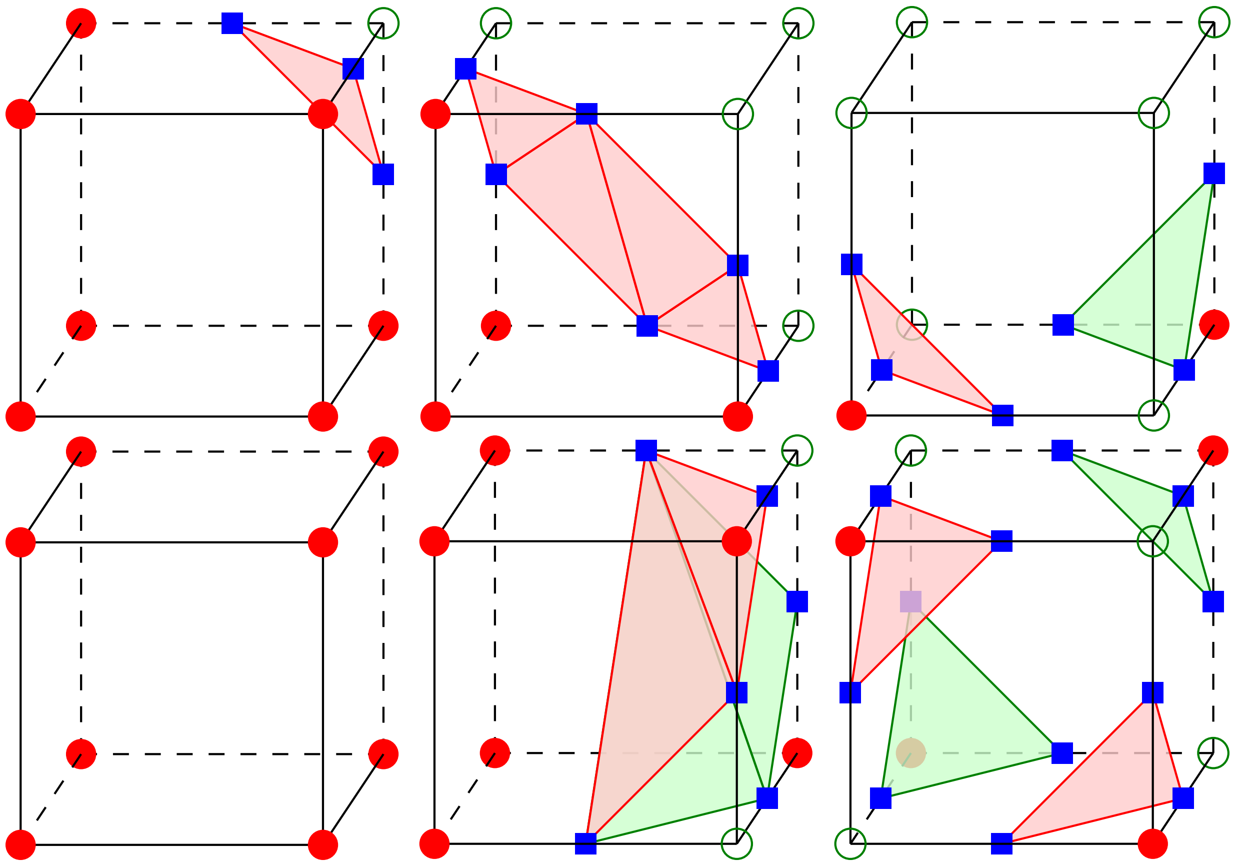
Problem 9 (2 points)

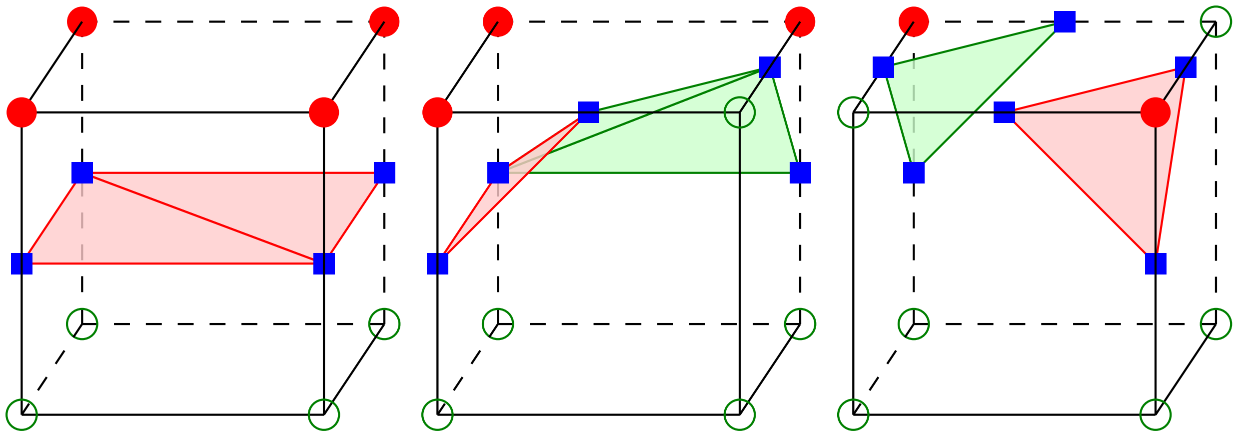
When viewed from 20 cm away, the whiteboard appears white. When viewed from a distance of 20 m away, the whiteboard appears to be approximately the same color and brightness. In the second case, the observer is standing 100 times farther away. Why does the whiteboard not appear 10,000 times dimmer (essentially completely black)?

In any fixed area of your retina, the light reaching your eye from any patch of the board will indeed be reduced by a factor of 10,000. However, the area of the board that reaches that part of your retina is multiplied by a factor of 10,000. The result is that the brightness stays about the same. You do still receive 10,000 times less light since the board shines that light over an area of your retina 10,000 times smaller.

Problem 10 (4 points)

Construct a consistent marching cubes triangulation of the cubes shown below. The cubes should actually be touching, but they have been separated apart for clarity. Draw squares at the vertices of your triangles.





Problem 11 (4 points)

For each of the following, either (1) show that the statement is always true, or (2) provide an example where it is false.

(a) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(b) $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w}$

(c) $\mathbf{u} \times (\mathbf{u} \times \mathbf{w}) = \mathbf{0}$

(d) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{0}$

(e) $(\mathbf{u} \cdot \mathbf{v})\mathbf{u} = (\mathbf{u} \cdot \mathbf{u})\mathbf{v}$

(f) $\|\mathbf{u} \times \mathbf{v}\| = 1$ whenever $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$.

(g) If $\|a\mathbf{u}\| = \|\mathbf{u}\|$ then $a = 1$.

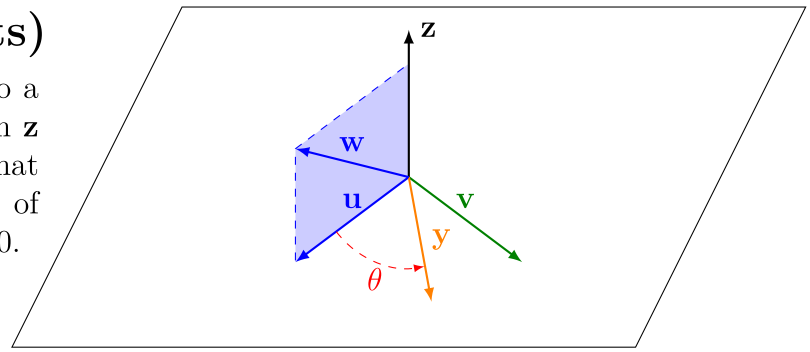
(h) If $\|\mathbf{u}\| = 0$ then $\mathbf{u} = \mathbf{0}$.

(a) true, (b) true, (c) false, (d) true, (e) false, (f) false, (g) false, (h) true.

In the following sequence of problems, we will work out formulas for 3D rotations about a given axis $\mathbf{z}$ by a given angle θ. You may assume $\|\mathbf{z}\| = 1$. We will call the rotation matrix $\mathbf{R}$; our goal is to derive a formula for $\mathbf{R}$. We will do this by considering its effects on an arbitrary vector $\mathbf{w}$. That is, we will work out a formula for $\mathbf{Rw}$ that depends only on $\mathbf{w}$, $\mathbf{z}$, and θ. *All of the parts below may be completed independently and in any order. You may assume the results of earlier problems when completing later ones. In particular, some parts ask you to compute new variables; you may use those variables in later parts, even if you have not computed them. It is important to read earlier problems, since they contain information you may need to complete later ones.*

Problem 12 (1 points)

We begin by decomposing $\mathbf{w}$ into a part $a\mathbf{z}$ along the axis of rotation $\mathbf{z}$ and a part $\mathbf{u}$ orthogonal to $\mathbf{z}$, so that $\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Find a in terms of $\mathbf{w}, \mathbf{z}, \theta$. You may assume $\mathbf{u} \cdot \mathbf{z} = 0$.



$\mathbf{w} = \mathbf{u} + a\mathbf{z}$. Using orthogonality and the fact that $\mathbf{z}$ is normalized, $\mathbf{w} \cdot \mathbf{z} = \mathbf{u} \cdot \mathbf{z} + a\mathbf{z} \cdot \mathbf{z} = a$.

Problem 13 (1 points)

Find $\mathbf{u}$ in terms of $\mathbf{w}, \mathbf{z}, \theta, a$.

$\mathbf{u} = \mathbf{w} - a\mathbf{z}$.

Problem 14 (1 points)

The plane shown has normal direction $\mathbf{z}$. Find another vector $\mathbf{v}$ in this plane, such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{z} \cdot \mathbf{v} = 0$. Select your vector so that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Your vector may depend on $\mathbf{w}, \mathbf{z}, \mathbf{u}, \theta, a$.

$\mathbf{v} = \mathbf{z} \times \mathbf{u}$.

Problem 15 (1 points)

Vectors in this plane will rotate by the full angle θ. Let $\mathbf{y} = \mathbf{R}\mathbf{u}$ be the vector that will be obtained by rotating $\mathbf{u}$ around the axis $\mathbf{z}$ by angle θ. Since this vector is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, we can write $\mathbf{y} = b\mathbf{u} + c\mathbf{v}$. Since rotations do not change lengths, we also know $\|\mathbf{y}\| = \|\mathbf{u}\|$. Find b in terms of θ. (Partial credit if you express in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a$, but if you have done this correctly, only θ will remain after simplification.)

$$\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{u} &= b\mathbf{u} \cdot \mathbf{u} + c\mathbf{v} \cdot \mathbf{u} \\
\|\mathbf{y}\|\|\mathbf{u}\| \cos\theta &= b\|\mathbf{u}\|^2 \\
b &= \cos\theta
\end{aligned}$$

Problem 16 (1 points)

Find c. There are two solutions; either is fine. Full credit if c is expressed in terms of b, θ only; partial credit if in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \theta, a, b$.

$$\begin{aligned}
\mathbf{y} &= b\mathbf{u} + c\mathbf{v} \\
\mathbf{y} \cdot \mathbf{y} &= (b\mathbf{u} + c\mathbf{v}) \cdot (b\mathbf{u} + c\mathbf{v}) \\
\|\mathbf{y}\|^2 &= b^2\mathbf{u} \cdot \mathbf{u} + 2bc\mathbf{u} \cdot \mathbf{v} + c^2\mathbf{v} \cdot \mathbf{v} \\
\|\mathbf{y}\|^2 &= b^2\|\mathbf{u}\|^2 + c^2\|\mathbf{v}\|^2 \\
c^2 &= 1 - b^2 \\
c &= \pm\sqrt{1 - b^2} = \pm\sin\theta
\end{aligned}$$

Problem 17 (1 points)

Let $\mathbf{q} = \mathbf{R}\mathbf{z}$. Find $\mathbf{q}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \theta, a, b, c$.
Rotations do not change vectors along the axis of rotation, so $\mathbf{q} = \mathbf{z}$.

Problem 18 (1 points)

Find $\mathbf{R}\mathbf{w}$ in terms of $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{q}, \theta, a, b, c$.
$\mathbf{R}\mathbf{w} = \mathbf{R}\mathbf{u} + a\mathbf{R}\mathbf{z} = \mathbf{y} + a\mathbf{q}$

Problem 19 (1 points)

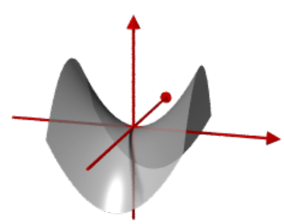
Find a matrix $\mathbf{M}$ such that $\mathbf{M}\mathbf{s} = (\mathbf{s} \cdot \mathbf{z})\mathbf{z}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{M}$ may only depend on the vector $\mathbf{z}$.
$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

Problem 20 (1 points)

Find a matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{s} = \mathbf{z} \times \mathbf{s}$ for any vector $\mathbf{s}$. Your matrix $\mathbf{A}$ may only depend on the vector $\mathbf{z}$. (You will need to write out $\mathbf{A}$ in terms of the components of $\mathbf{z}$.)

$\mathbf{M} = \mathbf{z}\mathbf{z}^T$.

The next three problems refer to the hyperbolic paraboloid defined by the implicit function $f(x, y, z) = x^2 - y^2 - z$. You may assume that $f(x, y, z) > 0$ corresponds to *outside* of the surface. The surface is illustrated at right. These problems also refer to the *ray* defined by $g(t) = \begin{pmatrix} 2t - 1 \\ -t - 1 \\ -2t - 1 \end{pmatrix}$, where $t \geq 0$. All of these problems are independent (you can solve them in any order, even if you have not solved earlier ones).



Problem 21 (2 points)

What are the direction and endpoint of the ray?

Endpoint is $g(0) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$. Direction is $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$. (Don't forget to normalize!)

Problem 22 (2 points)

Compute the closest intersection *location* of the surface with the ray. [Hint: the answer should not contain square roots; if it does not, check your math.]

Plugging $g(t)$ into $f(x, y, z)$ for the cone we get $(2t-1)^2-(-t-1)^2-(-2t-1) = 3t^2-4t+1 = (3t-1)(t-1)$. Thus, $t = \frac{1}{3}, 1$. The first is closer, so the closest intersection location is $g(\frac{1}{3}) = (-\frac{1}{3}, -\frac{4}{3}, -\frac{5}{3})$. The farther intersection location is at $g(1) = (1, -2, -3)$.

Problem 23 (2 points)

What is the normal direction at an arbitrary point (x, y, z) lying on the surface? Don't worry about whether the normal points inwards or outwards.

$$\begin{aligned} f &= x^2 - y^2 - z \\ \nabla f &= \begin{pmatrix} 2x \\ -2y \\ -1 \end{pmatrix} \\ \|\nabla f\| &= \sqrt{4x^2 + 4y^2 + 1} \\ n = \frac{\nabla f}{\|\nabla f\|} &= \frac{1}{\sqrt{4x^2 + 4y^2 + 1}} \begin{pmatrix} 2x \\ -2y \\ -1 \end{pmatrix} \end{aligned}$$

An alternative strategy is to parameterize the surface, such as with (x, y) and $\mathbf{w} = (x, y, z) =$

$(x, y, x^2 - y^2)$.

$$\mathbf{w} = \begin{pmatrix} x \\ y \\ x^2 - y^2 \end{pmatrix} \quad \mathbf{w}_x = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \quad \mathbf{w}_y = \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} \quad \mathbf{w}_x \times \mathbf{w}_y = \begin{pmatrix} (0)(-2y) - (2x)(1) \\ (2x)(0) - (1)(-2y) \\ (1)(1) - (0)(0) \end{pmatrix} = \begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix}$$

This agrees, up to sign, with what we obtained the original way.

Problem 24 (3 points)

For some of these questions, you will be asked to make calculations without using a calculator. You are provided with a trig table on the right, which is in degrees. You will get credit for your calculations as long as you are within 10% *of the correct value.* Snell's law $\mathbf{n}_i \sin\theta_i = \mathbf{n}_o \sin\theta_o$ relates the incoming angle θ_i and outgoing angle θ_o for rays passing from one medium to another (e.g., from air to glass). We will assume that our materials have $\mathbf{n}_i = 1.5$ and $\mathbf{n}_o = 1$. For each of the following incoming angles, estimate the angle of the **reflected ray** and the **transmitted ray**.

(a) $\theta_i = 10°$
(b) $\theta_i = 30°$
(c) $\theta_i = 50°$

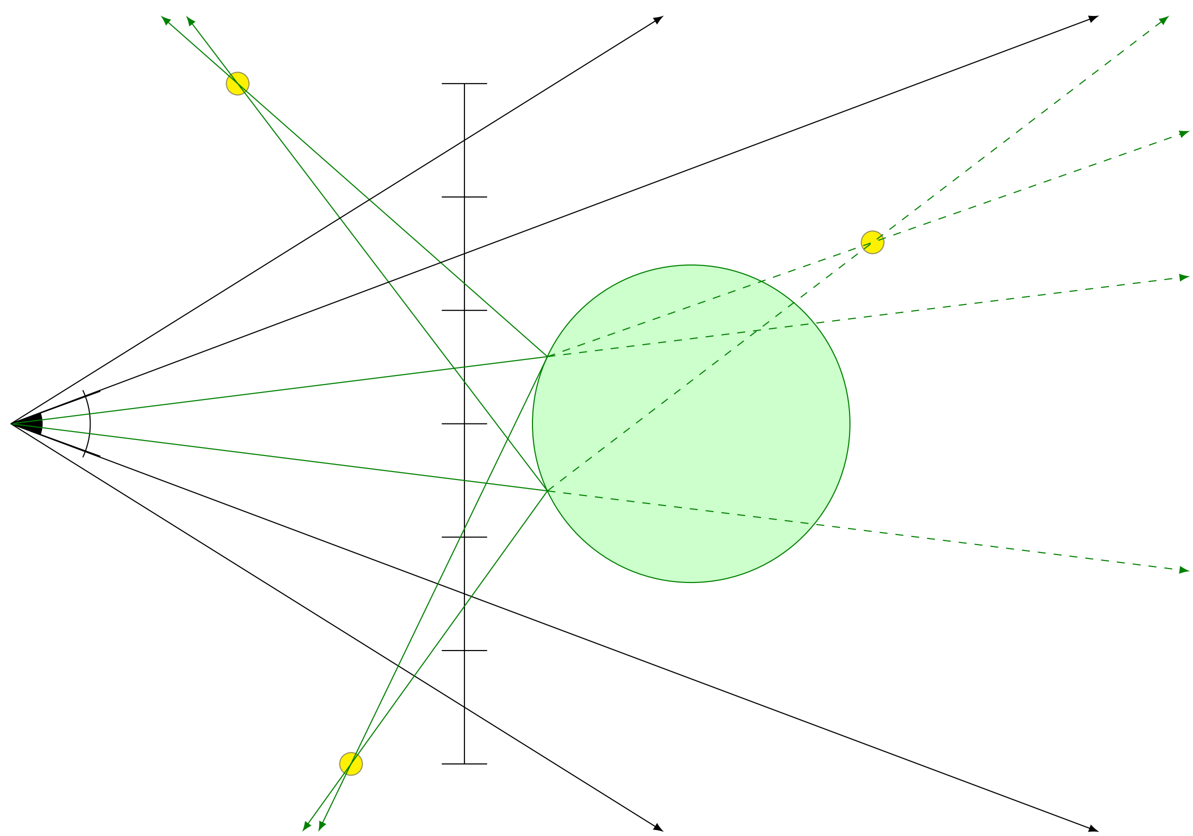
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0.00	1.00	0.00
10	0.17	0.98	0.18
20	0.34	0.94	0.36
30	0.50	0.87	0.58
40	0.64	0.77	0.84
50	0.77	0.64	1.19
60	0.87	0.50	1.73
70	0.94	0.34	2.75
80	0.98	0.17	5.67
90	1.00	0.00	∞

(a) reflected is 10°, transmitted is 15.1°.
(b) reflected is 30°, transmitted is 48.6°.
(c) reflected is 50°, and no transmitted ray exists. Complete internal reflection.

In the raytracing problems below, green objects are wood, red objects are reflective, and blue objects are transparent. The scenes are in 2D with a 1D image. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing each scene. Use a maximum recursion depth of 6. (Don't worry about precisely what counts as depth 6; I just care that recursion is being performed correctly when necessary and that important rays are not missing. There are no more than 30 rays in the "exact" solution.)

Problem 25 (4 points)

Note that this image has six pixels.



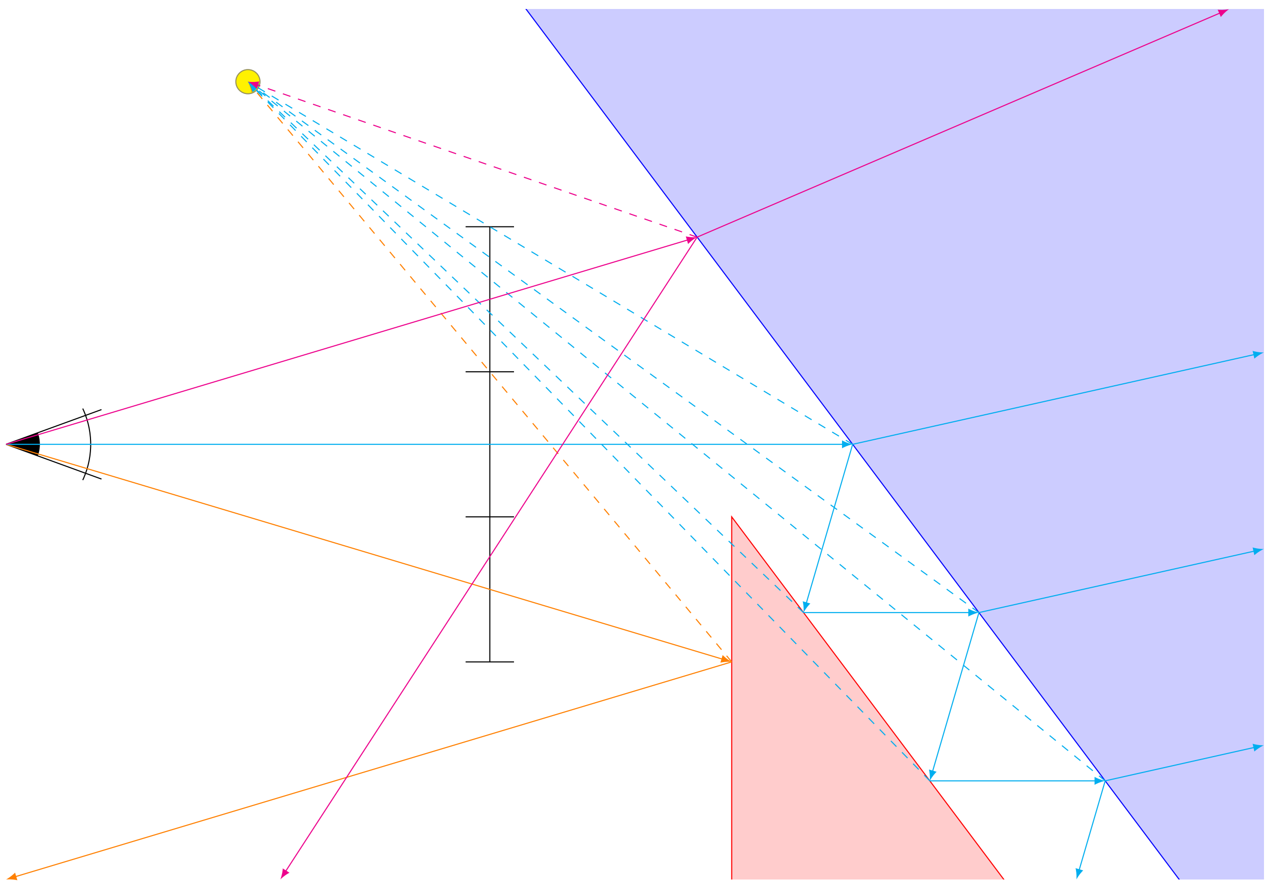
Problem 26 (2 points)

Why does a bump-mapped sphere cast a shadow that does not look bumpy?

Bump mapping only changes the color computation of the shader, not the physical geometry of the sphere. Since the sphere is geometrically smooth, so too are shadows and the silhouette. The intersection routines are not aware of the bump map.

Problem 27 (4 points)

For this problem, assume a recursion depth of 6. (If you have recursion depth 5-7, it is fine.)



Problem 28 (2 points)

What geometrical transformation does this 2D homogeneous transformation matrix correspond to? $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

It scales by 1/3 in the x direction and 2/3 in the y direction. Note that it scales the w by 3, so that when the perspective divide is performed, this 3 is divided from the x and the y.

[1]

[1] Total points: 59