CS 130, Final

Solutions

Read the entire exam before beginning. **Manage your time carefully.** This exam has 60 points; you need 50 to get full credit. Additional points are extra credit. 60 points → 3.0 min/point. 50 points → 3.6 min/point.

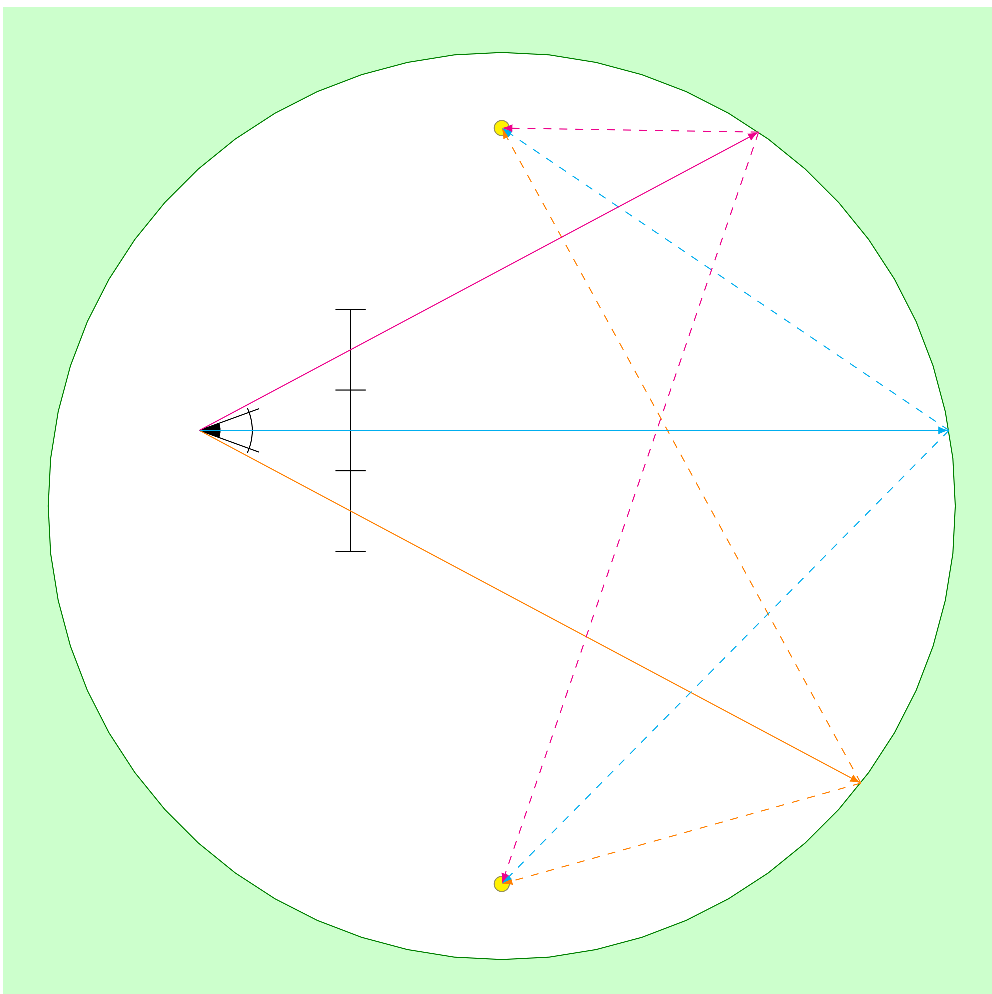
Problem 1 (5 points)

Identify the maximum and minimum possible values for each of the expressions in the table. $\mathbf{u}$ and $\mathbf{v}$ are **normalized** 3D vectors. $\mathbf{w}$ is also a 3D vector, but it need not be normalized. x and y are real-valued scalars. Write $-\infty$ if there is no minimum and ∞ if there is no maximum. No justification is required, but only answers in the table count.

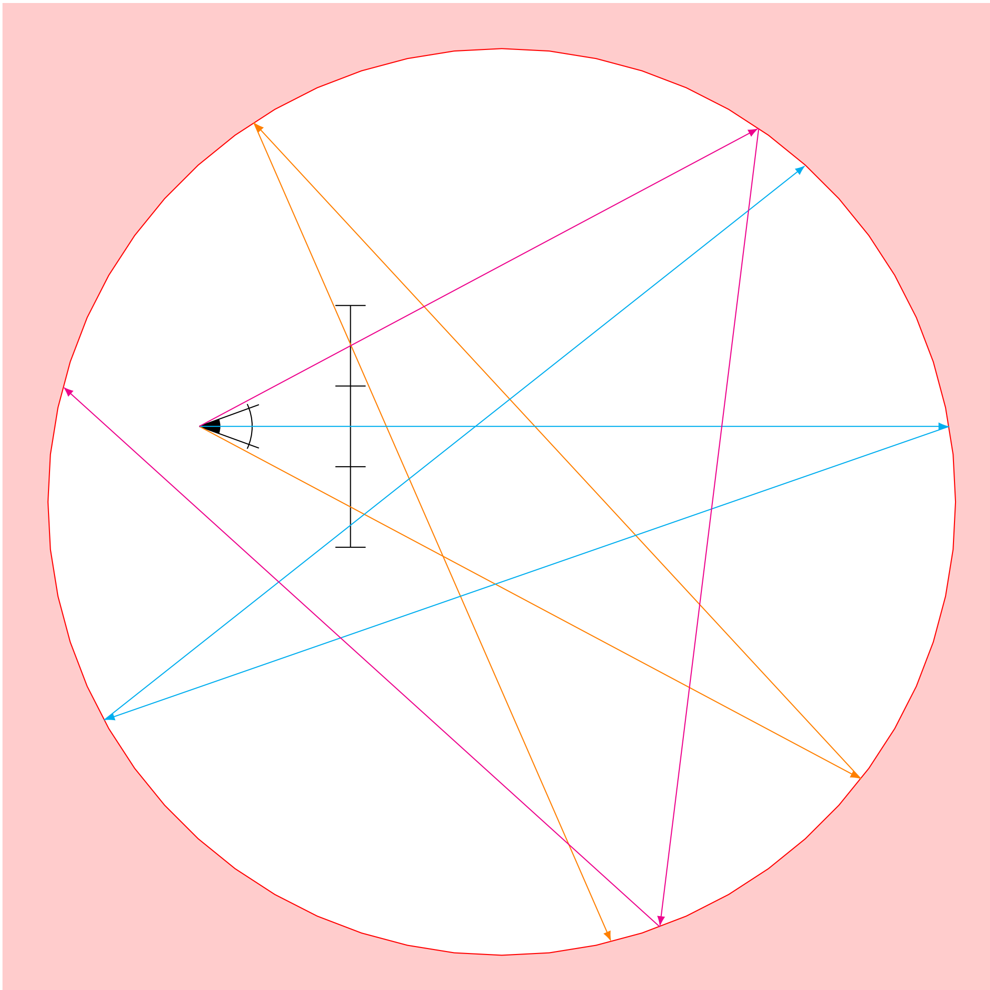
expression	minimum	maximum	expression	minimum	maximum
$x+1$	$-\infty$	∞	x^2+1	1	∞
$\\|\mathbf{u}\\|$	1	2	$\\|\mathbf{w}\\|$	0	∞
$1-\mathbf{u}\cdot\mathbf{v}$	0	1	$\frac{\\|x\mathbf{u}\\|}{x} \quad (x \neq 0)$	-1	1
$(\mathbf{w}\cdot\mathbf{w})-(\mathbf{w}\cdot\mathbf{u})^2$	0	∞	$(\mathbf{u}\cdot\mathbf{u})-(\mathbf{w}\cdot\mathbf{u})^2$	$-\infty$	1
$\\|\mathbf{u}+4\mathbf{v}\\|$	3	5	$\\|\mathbf{u}\times\mathbf{v}\\|^2+(\mathbf{u}\cdot\mathbf{v})^2$	1	1
x^2+xy+y^2	0	∞	$x^2+3xy+y^2$	$-\infty$	∞
x^2+x+1	$\frac{3}{4}$	∞	x^2-x+1	$\frac{3}{4}$	∞

In the raytracing problems below, green objects are wood, red objects are reflective, and blue objects are transparent. The scenes are in 2D with a 1D image. yellow circles are point lights; the ray tracer supports shadows. Draw all of the rays that would be cast while raytracing each scene. Use a maximum recursion depth of 3. (Don't worry about precisely what counts as depth 3; I just care that recursion is being performed correctly when necessary and that important rays are not missing. There are no more than 30 rays in the "exact" solution.)

Problem 2 (4 points)



Problem 3 (5 points)

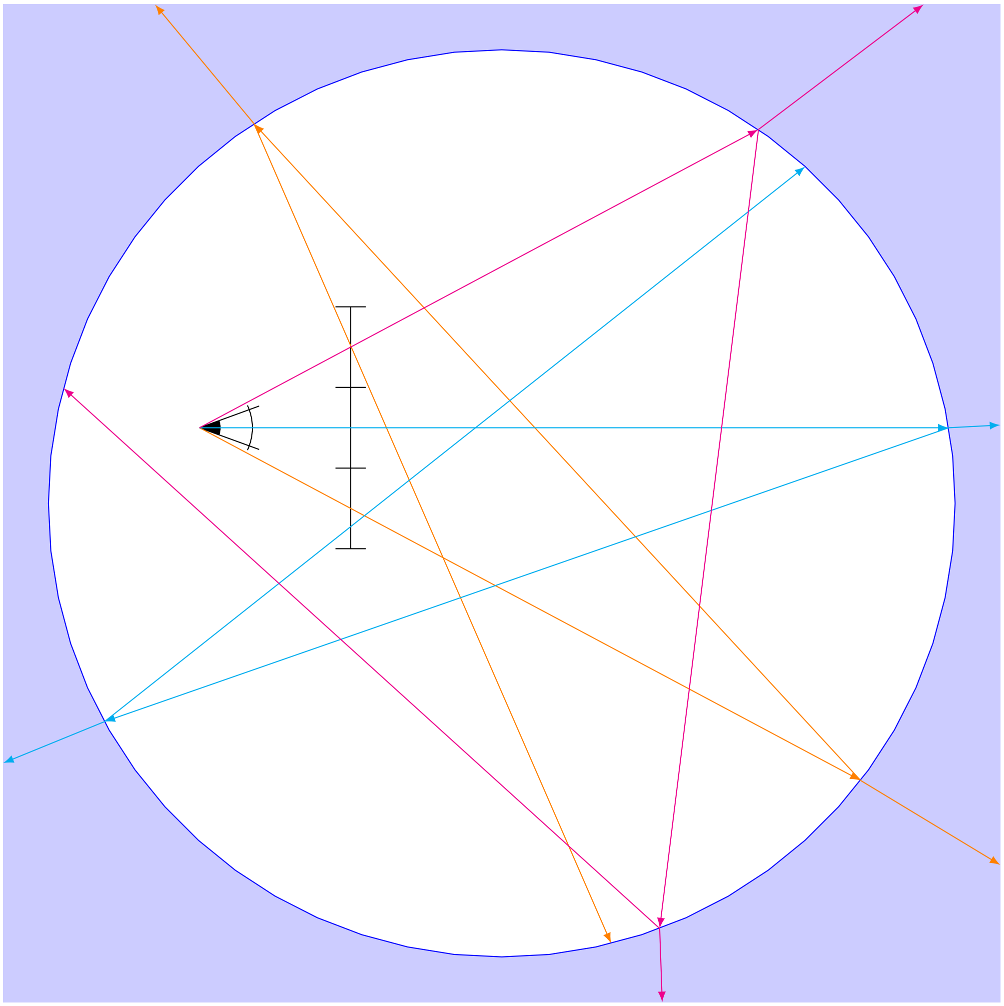


Problem 4 (2 points)

What sort of activity might one perform in a fragment shader? Be specific.

This is normally where the color of a pixel is computed (eg, a phong shader). Texture mapping is also applied here.

Problem 5 (5 points)



Problem 6 (2 points)

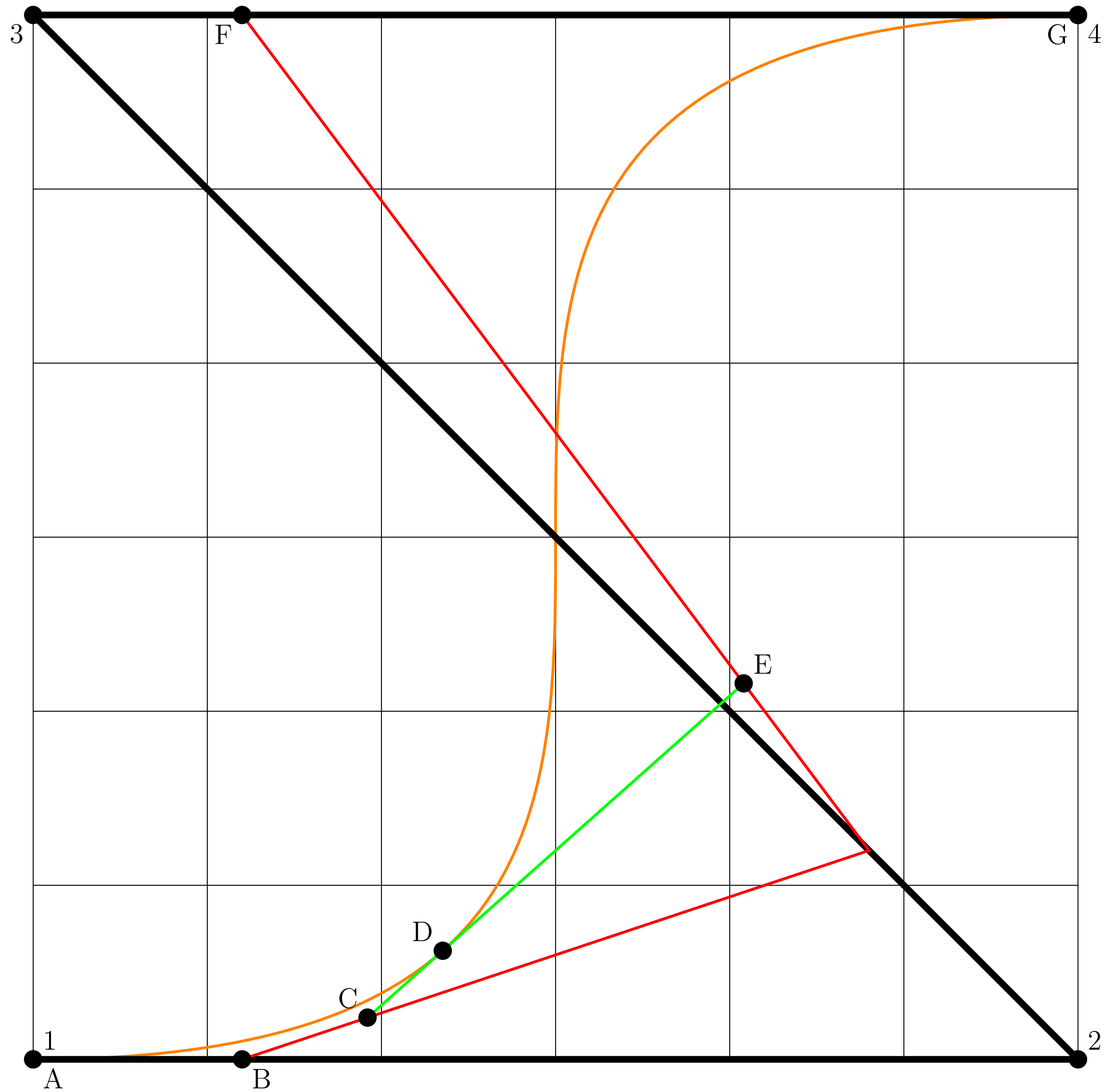
When working through 3D rotations, we defined a matrix $\mathbf{u}^*$ from a vector $\mathbf{u}$ such that matrix-vector multiplication was equivalent to cross product: $(\mathbf{u}^*)\mathbf{v} = \mathbf{u} \times \mathbf{v}$. Construct the matrix $\mathbf{u}^*$ explicitly from the components of $\mathbf{u} = \langle x, y, z \rangle$.

$$\mathbf{u}^* = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

Problem 7 (3 points)

Geometrically subdivide the Bézier curve given by the control points below and at $t = 0.2$

along the curve. Label the new control points A, B, C, . . . , G. (The two resulting Bézier curves should share their common point, so only 7 control points are required.)



Problem 8 (5 points)

Let $\mathbf{R}(\theta)$ be the 2×2 matrix that rotates counterclockwise by angle θ.
(a) Write out explicitly the matrix $\mathbf{R}(\theta)$.

(b) Find $\det(\mathbf{R}(\theta))$.

(c) Show that $\mathbf{R}(\theta)^T = \mathbf{R}(\theta)^{-1}$.

(d) Show that $\mathbf{R}(-\theta) = \mathbf{R}(\theta)^{-1}$.

(e) Show that $\mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\theta + \phi)$.

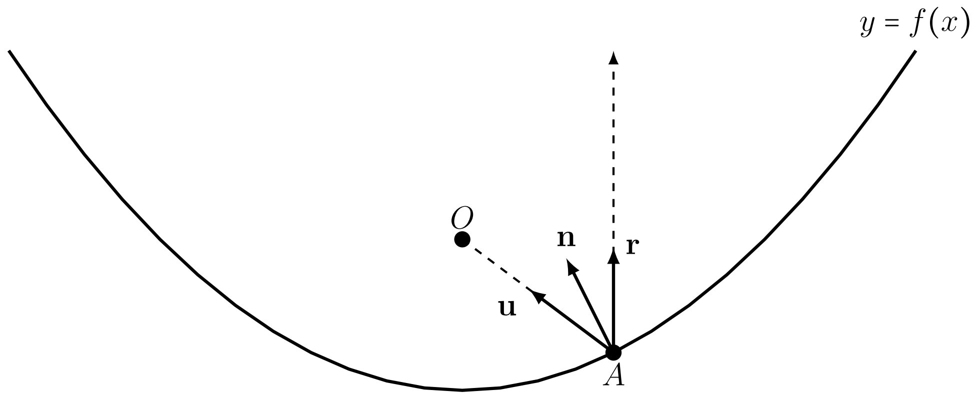
(a) $\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
(b) $\det(\mathbf{R}(\theta)) = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$
(c) $\mathbf{R}(\theta)^T\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \mathbf{I}$. Thus, $\mathbf{R}(\theta)^T = \mathbf{R}(\theta)^{-1}$.

(d) $\mathbf{R}(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathbf{R}(\theta)^T = \mathbf{R}(\theta)^{-1}$

(e)

$$\begin{aligned}
\mathbf{R}(\theta)\mathbf{R}(\phi) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \\
&= \mathbf{R}(\theta+\phi)
\end{aligned}$$

Mira specializes in making curved mirrors. In this case, she is making a mirror for a telescope. The observer sits at the location O, which is at the origin $(0,0)$. She wishes to design a special mirror, whose shape is given by the function $y = f(x)$. This mirror should have the property that when the observer at O looks at any point A on the surface of the mirror, the reflected ray travels vertically. This is useful when observing far-away objects like stars.



Problem 9 (2 points)

Let $A = (x, f(x))$ be an arbitrary point on the mirror, as shown. Find the direction $\mathbf{u}$ of the ray from O to A.

The direction is $\mathbf{u} = \frac{1}{\sqrt{x^2+f(x)^2}} \begin{pmatrix} -x \\ -f(x) \end{pmatrix}$.

Problem 10 (2 points)

Find the normal direction $\mathbf{n}$ of the mirror at A. Your normal direction should face upwards from the mirror.

One way to do this is to formulate an implicit function $g(x,y) = y - f(x)$, so that the normal direction is

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{1}{\sqrt{f'(x)^2 + 1}} \begin{pmatrix} -f'(x) \\ 1 \end{pmatrix}$$

Problem 11 (4 points)

We would like the reflected ray to have direction $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This places restrictions on what the function $f(x)$ might be. Find an equation that expresses the fact that the reflected ray will point in this direction. This equation will involve $f(x)$ and $f'(x)$. (Hint #1: computing the reflected ray directly as we did in class is very tedious; there is a simpler way to express this requirement. Hint #2: you should be able to manipulate your equation so that it is a fairly simple polynomial in terms of x, $f(x)$, and $f'(x)$. Doing so will make a later problem easier.)

Since the angles between $\mathbf{r}$ and $\mathbf{n}$ should equal the angle between $\mathbf{n}$ and $\mathbf{u}$, we can use a simple dot product.

$$\begin{aligned}
\mathbf{r} \cdot \mathbf{n} &= \mathbf{u} \cdot \mathbf{n} \\
\frac{1}{\sqrt{f'(x)^2 + 1}} &= \frac{x f'(x) - f(x)}{\sqrt{f'(x)^2 + 1}\sqrt{x^2 + f(x)^2}} \\
\sqrt{x^2 + f(x)^2} &= x f'(x) - f(x) \\
x^2 + f(x)^2 &= (x f'(x) - f(x))^2 \\
x^2 + f(x)^2 &= x^2 f'(x)^2 - 2x f(x) f'(x) + f(x)^2 \\
0 &= x f'(x)^2 - 2 f(x) f'(x) - x
\end{aligned}$$

Problem 12 (2 points)

It turns out that all suitable functions $f(x)$ are parabolas with the form $f(x) = ax^2 + bx + c$. Use the symmetry of the problem to solve for one of the coefficients.

We know that the shape should be symmetrical on $x \to -x$, which implies $b = 0$.

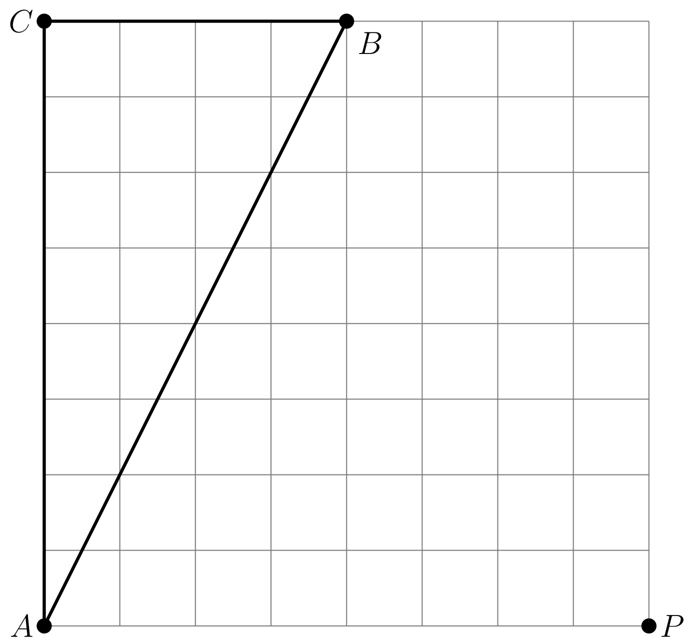
Problem 13 (2 points)

Use the equation your derived to solve for another of the coefficients of the parabola. You should be left with a one-parameter family of solutions.

$$
\begin{aligned}
0 &= xf'(x)^2 - 2f(x)f'(x) - x \\
0 &= x(2ax)^2 - 2(ax^2 + c)(2ax) - x \\
0 &= 4a^2x^3 - 4a^2x^3 - 4acx - x \\
0 &= (-4ac - 1)x \\
c &= \frac{-1}{4a} \\
f(x) &= ax^2 - \frac{1}{4a}
\end{aligned}
$$

Problem 14 (3 points)

Compute the barycentric weights of the point P.



Noting that barycentric coordinates are constant along lines parallel to the triangle edges and equally spaced, we immediately conclude that $\alpha = 1$ and $\beta = 2$, from which $\gamma = 1 - \alpha - \beta = -2$.

The coordinates are also straightforward to compute with triangle areas, since all of the triangles have a horizontal or vertical edge.

Problem 15 (2 points)

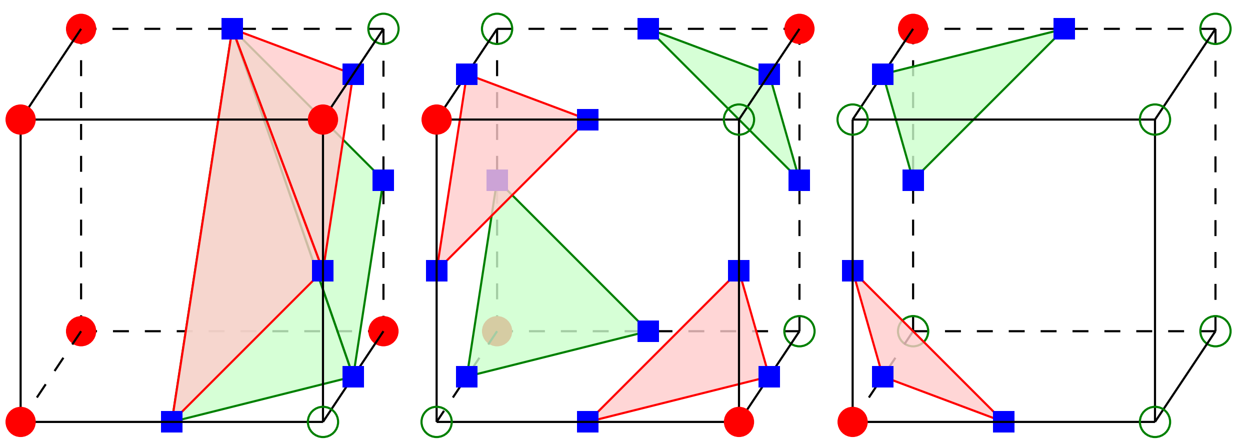
What is the angle between the vectors $\langle 1, 2\rangle$ and $\langle 3, 4\rangle$? (You do not need to simplify your answer.)

Using $u \cdot v = \|u\| \|v\| \cos\theta$ we have

$$\theta = \cos^{-1}\left(\frac{(1)(3) + (2)(4)}{\sqrt{1^2 + 2^2}\sqrt{3^2 + 4^2}}\right)$$
$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right)$$

Problem 16 (3 points)

Construct a consistent marching cubes triangulation of the cubes shown below. The cubes should actually be touching, but they have been separated apart for clarity. Draw squares at the vertices of your triangles.



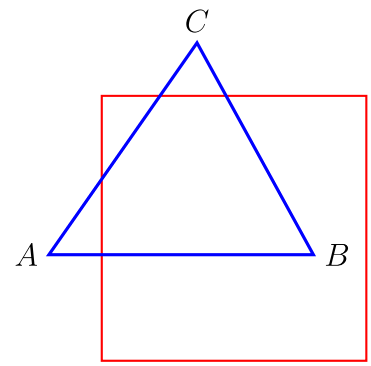
Problem 17 (2 points)

Which of the following types of transformations benefit from the use of homogeneous coordinates? No explanation is necessary, but no partial credit is possible. (a) Translation, (b) uniform scale, (c) non-uniform scale, (d) rotation, (e) reflection, (f) perspective projection, (g) orthographic transformation.

(a), (f), and (g).

Problem 18 (3 points)

Apply one of the clipping algorithms that we learned in class, step by step, to the triangle ABC. The clipping region is the red square. *Note that points are being awarded for demonstrating the steps of the algorithm. Merely showing what the results might look like does not score points.*



A correct solution for the triangle-based algorithm would clip the triangle against the edges of the square one by one. Triangles that get clipped into a quad should be divided back into two triangles.
A correct solution for the polygon-based algorithm would maintain a polygon (initially the triangle) and clip it against the edges of the square one by one.

Problem 19 (4 points)

In class, we learned to clip segments and triangles against a cube. In this problem, we will clip a line segment (endpoints A and B) against a sphere (center C radius r). Write a routine that (a) determines whether the line segment should be pruned and (b) if not, computes the clipped segment. You can use math notation in your pseudocode; it does not need to be C++ syntax.

First we need to work out the intersection of a line and a sphere. Let $\mathbf{x}$ be an intersection point.

$$\mathbf{x} = A + t(B - A) \qquad 0 \le t \le 1$$

$$\|\mathbf{x} - C\|^2 = r^2$$

$$0 = (A + t(B - A) - C) \cdot (A + t(B - A) - C) - r^2$$

$$0 = (\mathbf{u} + t\mathbf{w}) \cdot (\mathbf{u} + t\mathbf{w}) - r^2 \qquad \mathbf{u} = A - C \qquad \mathbf{w} = B - A$$

$$0 = (\mathbf{w} \cdot \mathbf{w})t^2 + 2(\mathbf{u} \cdot \mathbf{w})t + (\mathbf{u} \cdot \mathbf{u} - r^2)$$

$$0 = t^2 + 2bt + c \qquad b = \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \qquad c = \frac{\mathbf{u} \cdot \mathbf{u} - r^2}{\mathbf{w} \cdot \mathbf{w}}$$

$$t = -b \pm \sqrt{b^2 - c}$$

Next, we can sketch out our routine.

$$\mathbf{u} = A - C$$
$$\mathbf{w} = B - A$$
$$b = \frac{\mathbf{u} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$$
$$c = \frac{\mathbf{u} \cdot \mathbf{u} - r^2}{\mathbf{w} \cdot \mathbf{w}}$$
$$d = b^2 - c \qquad \text{If } d < 0\text{, prune out the segment and return.}$$
$$t_0 = -b - \sqrt{d} \qquad \text{If } t_0 > 1\text{, prune out the segment and return.}$$
$$t_1 = -b + \sqrt{d} \qquad \text{If } t_1 < 0\text{, prune out the segment and return.}$$
$$P = A + \max(t_0, 0)(B - A)$$
$$Q = A + \min(t_1, 1)(B - A)$$

Return segment PQ.

[1]

[1] Total points: 60