CS 230, Quiz 9

Solutions

You will have 8 minutes to complete this quiz. There are two problems; you only need to do **one** of them. No books, notes, or other aids are permitted.

Problem 1

The heat equation $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$ may be discretized with BTCS as $\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$. Show that the even-odd analysis from class does not lead to any restrictions on the time step size Δt for this discretization.

Let $u_{2i}^n = e^n$, $u_{2i+1}^n = o^n$, and $z^n = e^n - o^n$.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

Letting i be even or odd:

$$\frac{e^{n+1} - e^n}{\Delta t} = a\frac{o^{n+1} - 2e^{n+1} + o^{n+1}}{\Delta x^2} \qquad \text{and} \qquad \frac{o^{n+1} - o^n}{\Delta t} = a\frac{e^{n+1} - 2o^{n+1} + e^{n+1}}{\Delta x^2}$$

Subtract these:

$$\begin{aligned} \frac{z^{n+1} - z^n}{\Delta t} &= \frac{-4az^{n+1}}{\Delta x^2} \\ \left(1 + \frac{4a\Delta t}{\Delta x^2}\right)z^{n+1} &= z^n \\ z^{n+1} &= \frac{z^n}{1 + \frac{4a\Delta t}{\Delta x^2}} \end{aligned}$$

Since the denominator is greater than one, the difference is decreasing in magnitude. This is true for any $\Delta t > 0$, so we do not find any restrictions on the time step size using this approach.

Problem 2

Explain the difference between the two time derivatives $\frac{Du}{Dt}$ and $\frac{\partial u}{\partial t}$ and show how they are related.

The first time derivative is the change in acceleration experienced by an observer moving with the material (eg, looking at the speedometer in a car from the passenger seat). The partial derivative is the change in velocity observed in the material moving past a fixed location (eg, the observation made by a police officer with a radar gun). The relationship can be worked out by assuming that an observer is moving with a car at position $x_k(t)$ and velocity $v_k(t)$. The relationship is then

$$\frac{Du}{Dt} = \frac{d}{dt}u(x_k(t),t) = \frac{\partial u}{\partial t}(x_k(t),t) + \frac{\partial u}{\partial x}(x_k(t),t)\frac{d}{dt}x_k(t) = \frac{\partial u}{\partial t} + (\nabla u)\cdot u = \frac{\partial u}{\partial t} + (u \cdot \nabla)u$$