Gradient: $\nabla p = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$

Divergence: $\nabla \cdot \vec{u} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$ $\quad u = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$

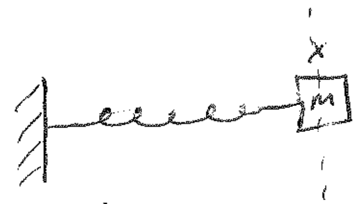
Laplacian: $\nabla^2 p = \nabla \cdot \nabla p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$

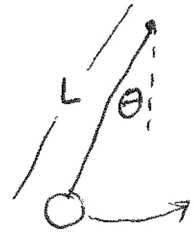
ODE:

$m\ddot{x} = -kx \leftarrow$ spring force

$ma = f$

$L\ddot{\theta} = -g \sin\theta$





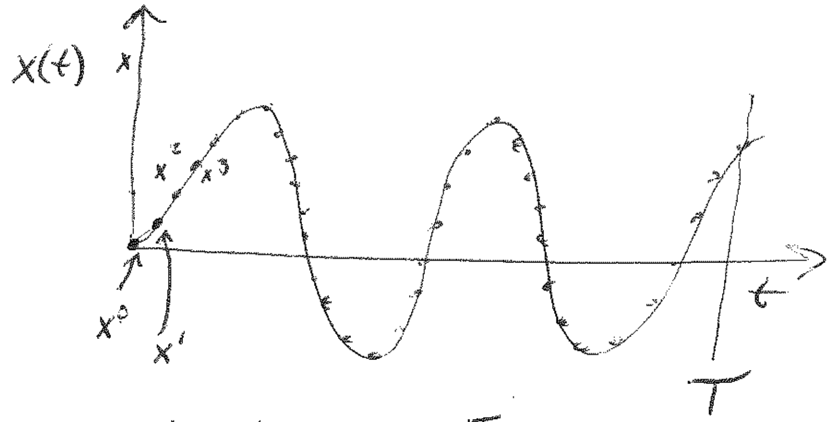
PDE:

Heat: $\frac{\partial u}{\partial t} - a\nabla^2 u = 0$

Waves: $\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0$

$$m\ddot{x} = -kx$$

$x(0) = x_0 \qquad v = \dot{x} \qquad v(0) = v_0$



$\underline{x^n} = x(n\,\Delta t)$ ↑ not exponent!

$t = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots, T$

$$\Delta t = \frac{T}{N} \leftarrow$$

$\underline{v^n} = v(n\,\Delta t)$

$x^0 = x_0 \qquad v^0 = v_0$

$m\dot{v} = -kx \qquad \dot{x} = v$

$$\dot{v} \approx \frac{v^{n+1} - v^n}{\Delta t} \rightarrow \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$$\dot{x} = f(x)$$

$$\frac{x^{n+1} - x^n}{\Delta t} = f(x^n)$$

$$m\,\frac{v^{n+1} - v^n}{\Delta t} = -kx^n \qquad \frac{x^{n+1} - x^n}{\Delta t} = v^n \leftarrow \text{forward Euler}$$

$$v^{n+1} = v^n - \frac{k\,\Delta t}{m}\,x^n \qquad x^{n+1} = x^n + \Delta t\,v^n$$

$$m\,\frac{v^{n+1} - v^n}{\Delta t} = -kx^{n+1} \qquad \frac{x^{n+1} - x^n}{\Delta t} = v^{n+1} \qquad \frac{x^{n+1} - x^n}{\Delta t} = f(x^{n+1})$$

Backward Euler

$$\frac{x^{n+1} - x^n}{\Delta t} = \frac{f(x^n) + f(x^{n+1})}{2}$$ Trapezoid rule

$$\frac{x^{n+1} - x^n}{\Delta t} = f\left(\frac{x^n + x^{n+1}}{2}\right)$$ Midpoint Rule

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$$\dot{x} = \lambda x$$

$$x^{n+1} = x^n + \Delta t \lambda x^n \equiv \frac{x^{n+1} - x^n}{\Delta t} = \lambda x^n$$

Forward Euler

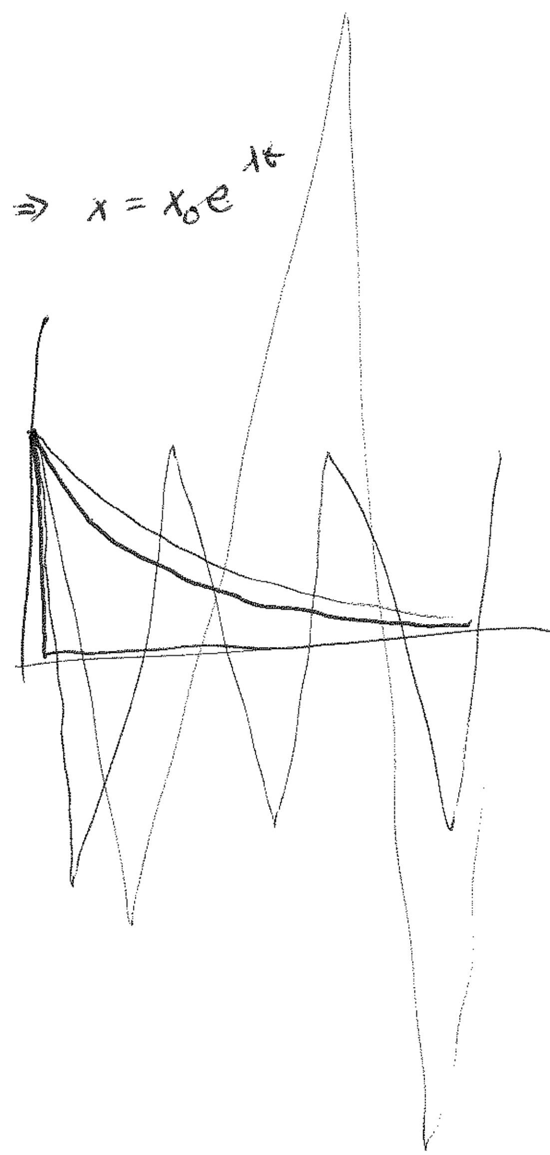
$$= x^n (1 + \Delta t \lambda)$$

$$x^n = x^0 \frac{(1 + \Delta t \lambda)^n}{1}$$

$\lambda < 0$

$\dot{x} = \lambda x \Rightarrow x = x_0 e^{\lambda t}$

$$|1 + \Delta t \lambda| < 1$$



$\Delta t = 3 \quad \lambda = -1$

$1 + \Delta t \lambda = -2$

$\Delta t = 2 \Rightarrow 1 + \Delta t \lambda = -1$

$\Delta t = 1 \Rightarrow 1 + \Delta t \lambda = 0$

$\Delta t = \frac{1}{2} \Rightarrow 1 + \Delta t \lambda = \frac{1}{2}$

Backward Euler

$$\frac{x^{n+1} - x^n}{\Delta t} = \lambda x^{n+1}$$

$$x^{n+1} - \Delta t \lambda x^{n+1} = x^n$$

$$x^{n+1} = \frac{x^n}{1 - \Delta t \lambda}$$

$$x^n = \frac{x^0}{(1 - \Delta t \lambda)^n}$$

$\lambda < 0 \quad \Delta t > 0 \qquad 1 - \Delta t \lambda > 1$

Unconditionally stable

$$\frac{x^{n+1} - x^n}{\Delta t} = f(x^{n+1}) \qquad f(x) = -\frac{g}{L} \sin x$$

$$x^{n+1} = x^n - \Delta t \frac{g}{L} \sin x^{n+1}$$

$$\frac{\partial u}{\partial t} - a\nabla^2 u = 0$$

1D $\quad u(x,t) \quad t \in [0,T] \quad x \in [0,1]$

Initial: $u(x,0) = f(x)$

Boundary Conditions

$u(0,t) = g(t)$

$u(1,t) = h(t)$