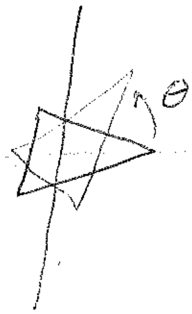
Rotations in 2D.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$



$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ca-sb & -(cb+sa) \\ sa+cb & ca-sb \end{pmatrix}$$

compose two rotations

note the similarity with complex numbers: $(c+si)(a+bi) = \cancel{(ca-sb)+(sa+cb)}$
$= (ca-sb)+(sa+cb)i$

$c^2+s^2=1 \Rightarrow |a+bi|^2=1$

$(\cos^2\theta+\sin^2\theta=1)$

* can represent 2D rotations as complex numbers with unit norm. complex multiply $\leftrightarrow$ matrix multiply

* matrix inverse $\rightarrow$ complex inverse $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = a-bi$ complex conjugate

* avoids duplication of arithematic in matrix multiply

* Dof count $z=c+si \rightarrow 2$ real dofs
$c^2+s^2=1 \rightarrow -1$ real constraint
$=1$ dof $\Rightarrow$ one rotational dof θ

note that applying rotation $(c+si)$ to vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is also multiplication: $(c+si)(x+yi) = (cx-sy)+(cy+sx)i \rightarrow \begin{pmatrix} cx-sy \\ cy+sx \end{pmatrix}$

Quaternions

extend complex numbers to $z = a + bi + cj + dk$

2D rotations = 2 dofs for complex − 1 dof for $|z|=1$ = 1 dof
3D rotations = 4 dofs for quaternion − 1 dof for $|z|=1$ = 3 dofs

Convenient to write $z = (s, \vec{v})$ $\quad \vec{v}$ is a 3-vector

$|z|^2 = a^2 + b^2 + c^2 + d^2 = s^2 + \|v\|^2$

how do we manipulate $z = (s, \vec{v})$ and $y = (r, \vec{u})$?

$y + z = (r+s, \vec{u} + \vec{v})$

$y - z = (r-s, \vec{u} - \vec{v})$

$kz = (ks, k\vec{v})$

$\bar{z} = (s, -\vec{v})$ conjugation (like for complex)

$yz = (rs - \vec{u}\cdot\vec{v}, r\vec{v} + s\vec{u} + \underbrace{\vec{u}\times\vec{v}}_{\neq \vec{v}\times\vec{u}})$ ← magical

, so $yz \neq zy$

$z\bar{z} = (s^2 + \vec{v}\cdot\vec{v}, s\vec{v} - s\vec{v} + \vec{v}\times\vec{v})$
$= (|z|^2, 0) = |z|^2$

thus, $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ just like complex numbers

note: if $|z| = 1$, then $z^{-1} = \bar{z}$.

Quaternions vs 3D rotations

$q = (s, \vec{v})$ represents our rotation

to rotate vector $\vec{\omega}$, compute

$$r = (0, \vec{\omega})$$

$$\underbrace{q\, r}_{a}\, q^{-1} = r'$$

$$a = (s, \vec{v})(0, \vec{\omega}) = (s \cdot 0 - \vec{v} \cdot \vec{\omega},\ s\vec{\omega} + 0\vec{v} + \vec{v} \times \vec{\omega}) = (-\vec{v} \cdot \vec{\omega},\ s\vec{\omega} + \vec{v} \times \vec{\omega})$$

$$q q^{-1} = \frac{1}{|q|^2}\, a\, \bar{q} = \frac{1}{|q|^2} (-\vec{v} \cdot \vec{\omega},\ s\vec{\omega} + \vec{v} \times \vec{\omega})(s, -\vec{v})$$

$$= \frac{1}{|q|^2} \Big(\underbrace{-s(\vec{v} \cdot \vec{\omega}) - s\vec{\omega} \cdot (-\vec{v})}_{0}\ \underbrace{- (\vec{v} \times \vec{\omega}) \cdot (-\vec{v})}_{0},\ (-\vec{v})(-\vec{v} \cdot \vec{\omega})$$

$$+ s^2 \vec{\omega} + s\vec{v} \times \vec{\omega} + (s\vec{\omega} + \vec{v} \times \vec{\omega}) \times (-\vec{v}) \Big)$$

$$= \frac{1}{|q|^2} \left(0,\ s^2 \vec{\omega} + 2s\vec{v} \times \vec{\omega} - (\vec{v} \cdot \vec{v})\vec{\omega} + 2(\vec{v} \cdot \vec{\omega})\vec{v}\right)$$

$$= \frac{1}{|q|^2} \left(0,\ (s^2 - \vec{v} \cdot \vec{v})\vec{\omega} + (2s)\vec{v} \times \vec{\omega} + 2(\vec{v} \cdot \vec{\omega})\vec{v}\right)$$

$$= r'$$

note: $(\vec{v} \times \vec{\omega}) \times \vec{v} = (\vec{v} \cdot \vec{v})\vec{\omega} - (\vec{v} \cdot \vec{\omega})\vec{v}$

assume $|q| = 1$

then

$$\vec{\omega}' = \underbrace{(s^2 - \vec{v} \cdot \vec{v})}_{a}\vec{\omega} + \underbrace{2s}_{b}\, \vec{v} \times \vec{\omega} + \underbrace{2(\vec{v} \cdot \vec{\omega})}_{c}\vec{v} = a\vec{\omega} + b\, v \times \omega + c\vec{v}$$

$$\|\vec{\omega}'\|^2 = (a\vec{\omega} + b\, \vec{v} \times \vec{\omega} + c\vec{v}) \cdot (a\vec{\omega} + b\, \vec{v} \times \vec{\omega} + c\vec{v})$$

$$= a^2 \vec{\omega} \cdot \vec{\omega} + b^2 \|v \times \omega\|^2 + c^2 \|\vec{v}\|^2 + 2ac\, \vec{\omega} \cdot \vec{v}$$

$$= s^4 \|\omega\|^2 - 2s^2 \|v\|^2 \|\omega\|^2 + \|v\|^4 \|\omega\|^2 + 4s^2 \|v\|^2 \|\omega\|^2 - 4s^2 (v \cdot \omega)^2 + 4(s^2 - \|v\|^2)(v \cdot \omega)^2$$

$=$

$\|v \times \omega\| = \|v\|\ \|\omega\| \sin\theta$

$v \cdot \omega = \|v\|\ \|\omega\| \cos\theta$

$\|v \times \omega\|^2 = \|v\|^2 \|\omega\|^2 - (v \cdot \omega)^2$

$\|\vec{\omega}'\|^2 = |r'|^2 = |q r q^{-1}|^2 = |q|^2 |r|^2 |q^{-1}|^2 = |r|^2 = \|\vec{\omega}\|^2$

note: $|q|\,|q^{-1}| = |q q^{-1}| = |1| = 1$

$r' = (0, \vec{\omega}')$
$r = (0, \vec{\omega})$

thus, transforming with q in this way preserves distance

note: $\|u+v\| - \|u-v\| = u \cdot u + 2u \cdot v + v \cdot v - u \cdot u + 2u \cdot v - v \cdot v = 4u \cdot v$

preserving length $\Rightarrow$ preserve dot product $\Rightarrow$ preserve angles

thus, we have rotation. (or reflection possibly).

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note: $\vec{\omega} = a\vec{v} \qquad q = (s, \vec{v})$

$r = (0, \vec{\omega}) = (0, a\vec{v})$

$(s, \vec{v})(0, a\vec{v}) = (-a\vec{v} \cdot \vec{v},\ as\vec{v} + 0 + 0)$

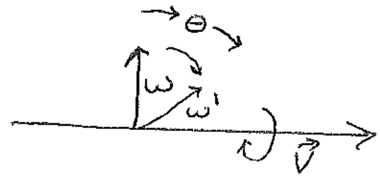
$(-a\vec{v} \cdot \vec{v},\ as\vec{v})(s, -\vec{v}) = (\underbrace{-sa\vec{v} \cdot \vec{v} + as\vec{v} \cdot \vec{v}}_{0},\ a(v \cdot v)v + as^2 v + 0)$

$= a\underbrace{(s^2 + v \cdot v)}_{|q|^2 = 1}\vec{v} = a\vec{v} = \vec{\omega}$

thus the rotation leaves $\vec{v}$ alone $\rightarrow$ rotation about $\vec{v}$.

$\vec{\omega} \cdot \vec{v} = 0 \qquad \|\vec{\omega}\| = 1$

then $\omega \cdot \omega' = \cos\theta$



$q = (s, \vec{v})$

$r = (0, \vec{\omega})$

$r' = (0, \vec{\omega}')$ $\qquad$ note: $|r| = \|\vec{\omega}\| = 1$

$\qquad\qquad\qquad\qquad\qquad\quad |r'| = \|\vec{\omega}'\| = 1$

$r^{-1} r' = (0, -\vec{\omega})(0, \vec{\omega}') = (\vec{\omega} \cdot \vec{\omega}', -\vec{\omega} \times \vec{\omega}')$

$= \bar{r} r'$

$r^{-1} r' = r^{-1} q r q^{-1}$

$$\vec{\omega}' = (s^2 - \vec{v} \cdot \vec{v})\vec{\omega} + 2s\, \vec{v} \times \vec{\omega} + \underbrace{2(\vec{v} \cdot \vec{\omega})\vec{v}}_{0}$$

$$\vec{\omega} \cdot \vec{\omega}' = (s^2 - \vec{v}\vec{v})\underbrace{\|\vec{\omega}\|^2}_{1} + \underbrace{2s(\vec{v} \times \vec{\omega}) \cdot \vec{\omega}}_{0}$$

$= s^2 - \vec{v}\vec{v}$

$= s^2 - t^2 = \cos\theta \qquad q = (s, t\vec{u}) \qquad \|\vec{u}\| = 1, \; \vec{v} = t\vec{u}$

$|q|^2 = s^2 + t^2 = 1$

$2s^2 = 1 + \cos\theta$

$s = \pm\sqrt{\frac{1 + \cos\theta}{2}} = \pm\cos\frac{\theta}{2}$

$\Rightarrow t = \pm\sin\frac{\theta}{2}$

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rotation by θ around $\vec{u}$ $\qquad \|\vec{u}\| = 1$

$\rightarrow q = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, \vec{u}\right)$

easy to write down given angle and axis,

Quaternion to rotation matrix

$q = (s, \vec{v})$

$$\vec{u}' = (s^2 - \vec{v}\cdot\vec{v})\vec{\omega} + 2s\,\vec{v}\times\vec{\omega} + 2(\vec{v}\cdot\vec{\omega})\vec{v}$$

$$= \underbrace{\left((s^2 - \vec{v}\cdot\vec{v})I + 2s\,\vec{v}^* + 2\vec{v}\vec{v}^T\right)}_{R}\vec{\omega}$$

rotation matrix

$$\underbrace{\vec{v}^*}_{\text{matrix}}\vec{\omega} = \vec{v}\times\vec{\omega}$$

$s = \cos\frac{\theta}{2}$ $\quad \|v\| = \sin\frac{\theta}{2}$ $\quad V = \sin\frac{\theta}{2}\vec{u}$

$$s^2 - \vec{v}\cdot\vec{v} = \left(\cos\frac{\theta}{2}\right)^2 - \left(\sin\frac{\theta}{2}\right)^2 = \cos\theta$$

$$2sv^* = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\,\vec{u}^* = \sin\theta\,\vec{u}^*$$

$$2\,\vec{v}\cdot\vec{v}^T = 2\left(\sin\frac{\theta}{2}\right)^2\vec{u}\vec{u}^T = (1-\cos\theta)\,\vec{u}\vec{u}^T$$

$$R = (\cos\theta)I + (\sin\theta)\vec{u}^* + (1-\cos\theta)\vec{u}\vec{u}^T$$

this is what you implimented for glRotate

Composition quaternions p, q

rotate by q, then by p

$$r' = qrq^{-1}$$

$$r'' = pr'p^{-1} = pqrq^{-1}p^{-1} = (pq)r(pq)^{-1}$$

rotation by pq

composition is quaternion multiplication