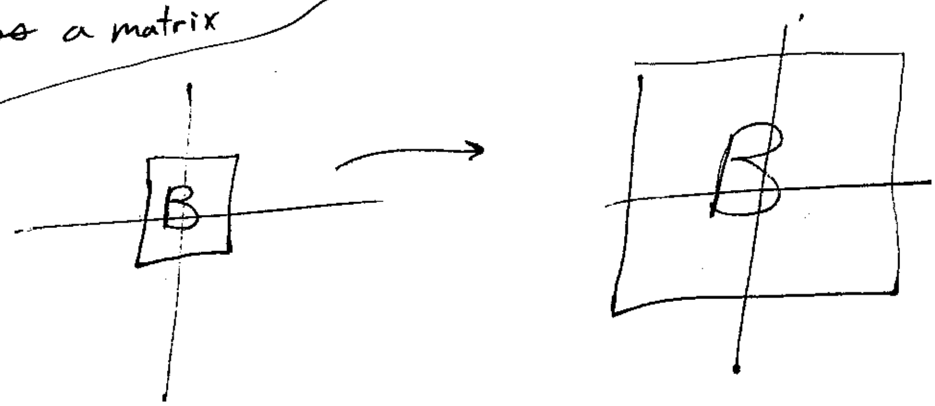
* transformations

Do the Matrix part of first lecture

→ 2D
- ↳ translation
- → rotation
- → uniform scale
- → non-uniform scale
- → shear

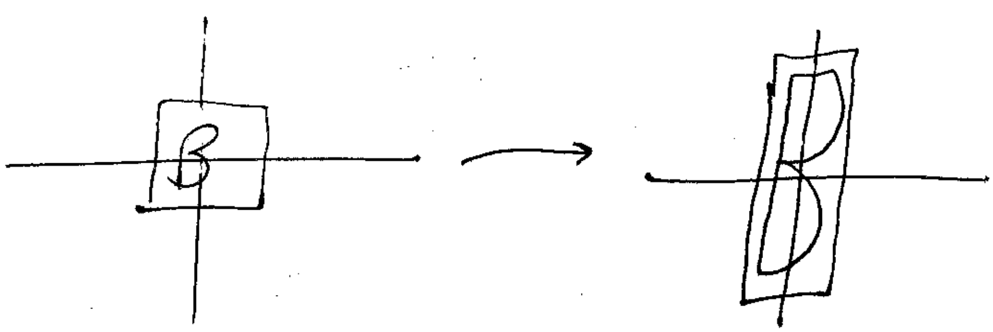
→ want to represent as a matrix

uniform scale



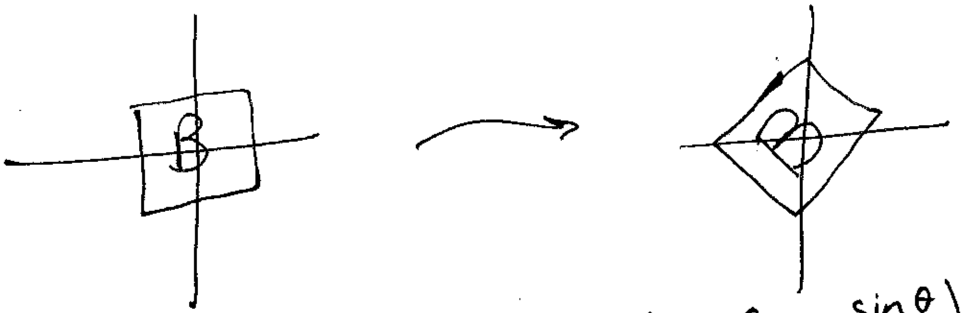
$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

non-uniform scale



$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} ax \\ by \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

rotation



Show preserves length

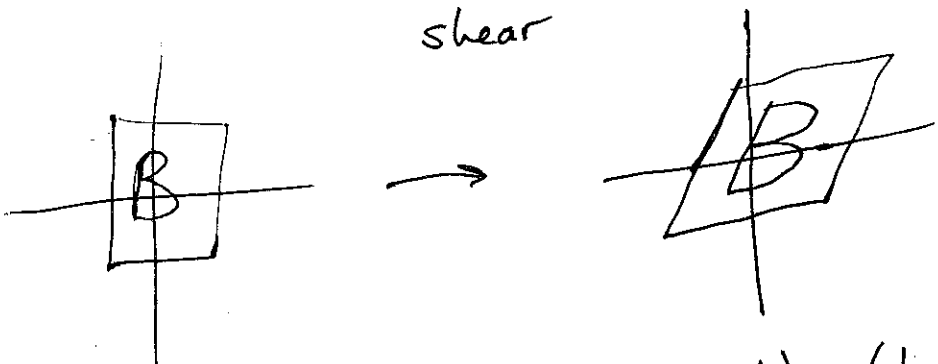
Show angle by dot product.

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

rotation CCW by θ

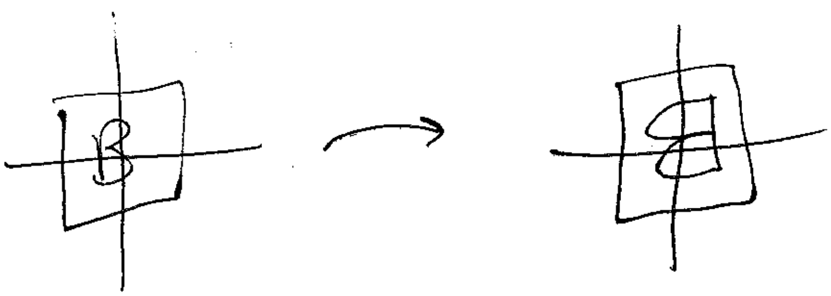
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

shear



$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x+ay \\ y \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

reflection



$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

translation

$\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} x+a \\ y+b \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

* does not work
* here is a trick

instead of $\begin{pmatrix} x \\ y \end{pmatrix}$ lets use $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

now $$\begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

this allows translation to be represented.

still have old stuff, too $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc.

Homogeneous Coordinates

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$$

how do we interpret $\begin{pmatrix} x \\ y \\ w \end{pmatrix}$?

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{x}{w} \\ \frac{y}{w} \end{pmatrix}$$

thus $\begin{pmatrix} x \\ y \\ w \end{pmatrix}$ and $\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix}$ are equivalent

* why this choice?
* ~~the~~ behaves nicely with respect to the transforms

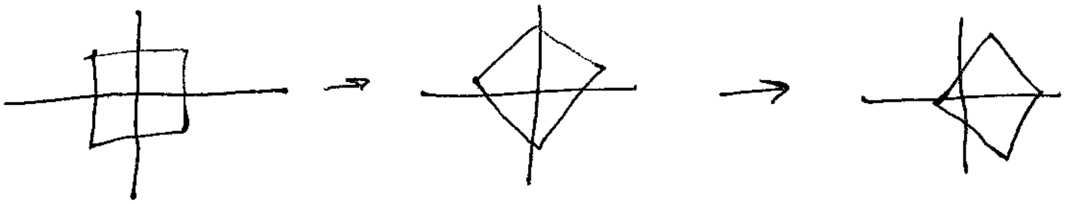
eg $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} cx \\ cy \\ c \end{pmatrix}}_{\text{equiv}} = \begin{pmatrix} cx+ca \\ cy+cb \\ c \end{pmatrix} = \underbrace{c\begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}}_{\text{equiv}}$$

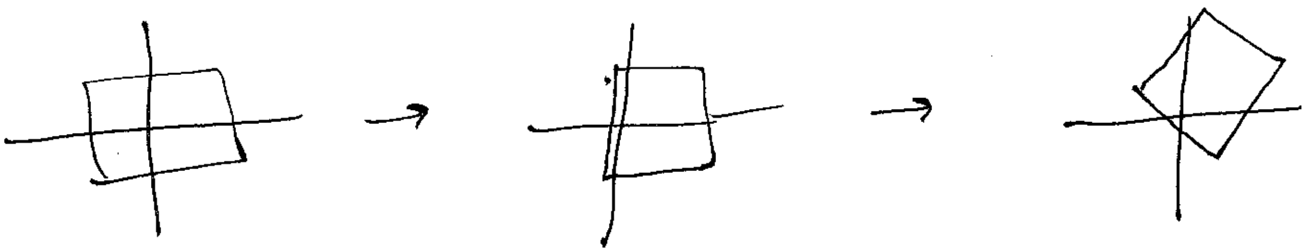
* will be very useful when we start doing ~~pers~~ perspective projections.
* ~~the~~ now basic transforms can all be represented as matrices.
* compose transforms = matrix multiply

Not commutative

① rotate, then translate



② translate, then rotate



different (rotate = rotate about origin)

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how about 3D?

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

use 4-vectors and 4×4 matrices

rotations $\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

etc.

Properties of Rotation Matrices

R = rotation

$y = Rx \qquad z = Rw$

must be true: $x \cdot w = y \cdot z$

$$\|x\| \, \|w\| \cos\theta = \|y\| \, \|z\| \cos\theta$$

preserves lengths

preserves angles

$\cancel{0 = x^T w - y^T z}$ $\qquad$ $0 = \cancel{z^T y - w}$

$$\begin{aligned}
&y^T z - x^T w \\
&= (Rx)^T (Rw) - x^T w \\
&= x^T R^T R w - x^T w \\
&= x^T (R^T R w - w) \qquad \text{any } x, w \\
&= x^T (R^T R - I) w
\end{aligned}$$

(eg $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$)

$\hookrightarrow R^T R = I$

$\hookrightarrow R^T = R^{-1}$

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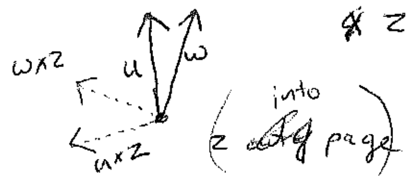
are all matrices with $R^T R = I$ rotations?

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Constructing a Rotation in 3D

given: u, w $\|u\| = \|w\|$

construct R (rotation) so that $w = Ru$



$z = u \times w$

$$\left.\begin{aligned} w &= Ru \\ z &= Rz \\ (w \times z) &= R(u \times z) \end{aligned}\right\} \rightarrow \begin{aligned} M &= \begin{pmatrix} w & z & w \times z \end{pmatrix} \\ N &= \begin{pmatrix} u & z & u \times z \end{pmatrix} \end{aligned}$$

$M = RN$

$$\boxed{R = MN^{-1}}$$