CS 130, Midterm 2

Solutions

Problem 1 (4 points)

Let $p = (r, \vec{w})$ and $q = (s, \vec{v})$ be two quaternions. An operation pq is said to commute if $pq = qp$.
(a) Show that quaternion multiplication does not, in general, commute.
(b) What conditions must p and q satisfy in order for their product to commute?
(c) The result of (b) has a simple geometric interpretation in terms of rotations. What is this interpretation?
(d) Show that q and $-q$ correspond to the rotation.

(a) For example, let $p = i$ and $q = j$. Then, $pq = ij = k$ and $qp = ji = -k$. Thus, quaternions fail to commute in at least some cases.
(b) $pq = (rs - \vec{w} \cdot \vec{v}, r\vec{v} + s\vec{w} + \vec{w} \times \vec{v})$. $qp = (sr - \vec{v} \cdot \vec{w}, s\vec{w} + r\vec{v} + \vec{v} \times \vec{w})$. These are equal only if $\vec{w} \times \vec{v} = \vec{v} \times \vec{w}$. That is, $\vec{w} \times \vec{v} = \vec{0}$. This only occurs if $\vec{v}$ and $\vec{w}$ are parallel.
(c) For a rotation, the vector part of the quaternion is the rotation axis. Rotations commute if (and only if) they are rotations about the same axis.
(d) Let $\vec{u}$ be a vector. Rotation by q is accomplished by $(0, \vec{u}') = q(0, \vec{u})q^{-1} = (-q)(0, \vec{u})(-q)^{-1}$. Thus, q and $-q$ are equivalent as rotations.

Problem 2 (4 points)

Let $p = (r, \vec{w})$ and $q = (s, \vec{v})$ be two quaternions. Let $m = pq$.
(a) Let a be a real number. Express a as a quaternion and show that $aq = qa$.
(b) Show that $\overline{m} = (\overline{q})(\overline{p})$.
(c) Use the definition $|q|^2 = s^2 + \|\vec{v}\|^2$ to show that $q\overline{q} = |q|^2$
(d) Using (a-c), show that $|pq|^2 = |p|^2|q|^2$ and thus $|pq| = |p||q|$.

(a) $q = a + 0i + 0j + 0k$ or $q = (a, \vec{0})$. Using the multiplication rule, $aq = (as, a\vec{v}) = qa$.
(b)

$$\begin{aligned} m = pq &= (rs - \vec{w} \cdot \vec{v}, r\vec{v} + s\vec{w} + \vec{w} \times \vec{v}) \\ \overline{m} &= (rs - \vec{w} \cdot \vec{v}, -r\vec{v} - s\vec{w} - \vec{w} \times \vec{v}) \\ &= (rs - (-\vec{v}) \cdot (-\vec{w}), r(-\vec{v}) + s(-\vec{w}) + (\vec{-v}) \times (-\vec{w})) \\ &= (s, -\vec{v})(r, -\vec{w}) \\ &= (\overline{q})(\overline{p}) \end{aligned}$$

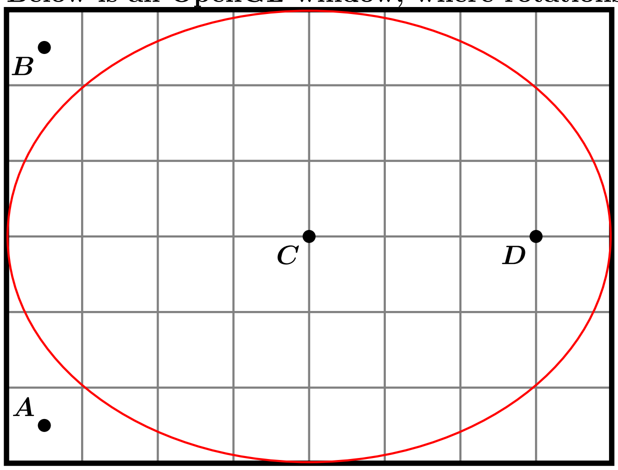
(c)

$$\begin{aligned} pq &= (rs - \vec{w} \cdot \vec{v}, r\vec{v} + s\vec{w} + \vec{w} \times \vec{v}) \\ p\overline{p} &= (r^2 + \vec{w} \cdot \vec{w}, -r\vec{w} + r\vec{w} - \vec{w} \times \vec{w}) \\ &= (r^2 + \|\vec{w}\|^2, \vec{0}) \\ &= r^2 + \|\vec{w}\|^2 \\ &= |p|^2 \end{aligned}$$

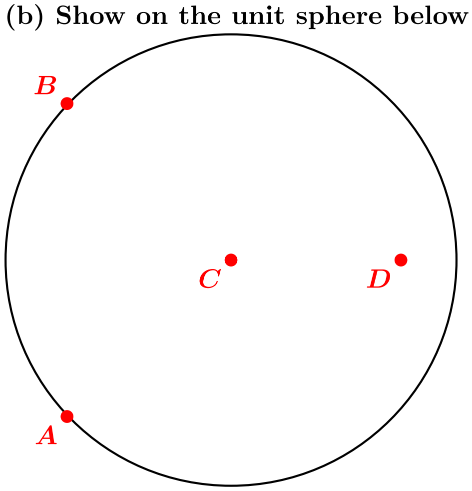
(d) $|pq|^2 = pq\overline{(pq)} = pq\overline{q}\,\overline{p} = p(q\overline{q})\overline{p} = p|q|^2\overline{p} = p\overline{p}|q|^2 = |p|^2|q|^2$.

Problem 3 (4 points)

Below is an OpenGL window, where rotations are being determined using arcball.



(a) Draw the arcball on the illustration above.
(b) Show on the unit sphere below where the points A, B, C, and D map onto the arcball.



(c) If the user clicks at A, drags to B along a straight line, then releases the mouse, what is the axis of rotation of the final rotation? (Estimate your answer. Don't try to compute it, normalize it, or get the sign right. Just get plausibly close. If you would rather, you can compute it exactly, but don't simplify your answer.)

(a) Shown red above. Note that it is an ellipse, not a circle.
(b) Shown red above.
(c) Note that both A and B will map to the xy plane, so the rotation will be in the plane. The rotation axis is $\langle 0, 0, 1 \rangle$.

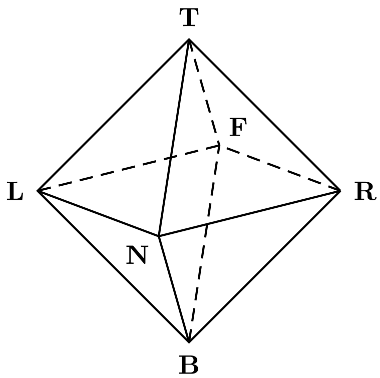
Problem 4 (4 points)

When viewed from 20 cm away, the whiteboard appears white. When viewed from a distance of 20 m away, the whiteboard appears to be approximately the same color and brightness. In the second case, the observer is standing 100 times farther away. Why does the whiteboard not appear 10,000 times dimmer (essentially completely black)?

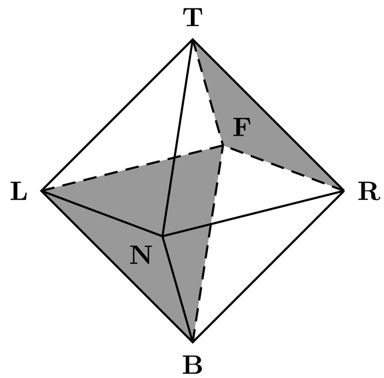
In any fixed area of your retina, the light reaching your eye from any patch of the board will indeed be reduced by a factor of 10,000. However, the area of the board that reaches that part of your retina is multiplied by a factor of 10,000. The result is that the brightness stays about the same. You do still receive 10,000 times less light since the board shines that light over an area of your retina 10,000 times smaller.

Problem 5 (4 points)

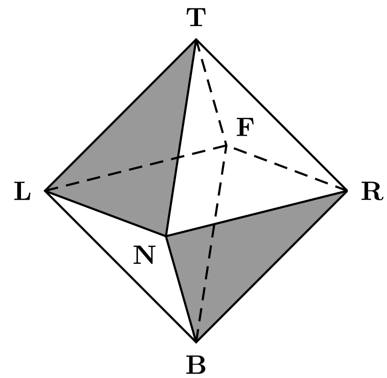
Below is shown an octahedron (eight triangles, six vertices). The original octahedron should be colored with a sort of checkerboard pattern, where triangles alternate light and dark. (The second figure shows the coloring of the back four triangles; the third figure shows the coloring of the front four triangles.)



original

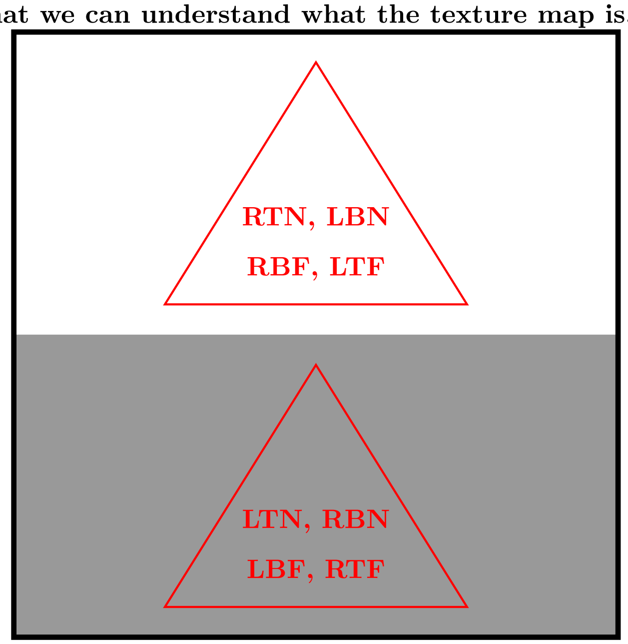


textured (back)



textured (front)

You are also given the texture image shown below. Construct a suitable texture map for the octahedron by drawing the triangles in the texture map and labeling them. (Labels like LBN, LTN, RBF, etc. are fine. Or you can label the vertices of each triangle. All we care is that we can understand what the texture map is.)



Many solutions are possible. All that really matters is that the dark triangles are placed in the dark part of the image and the light triangles in the light part. In the solution above, the dark and light triangles overlap, which is fine.

Problem 6 (4 points)

Let T be at triangle (in 3D) with vertices $\vec{a}$, $\vec{b}$, and $\vec{c}$. Let R be a ray $\vec{u} + t\vec{v}$. You may assume $\|\vec{c}\| = 1$ and that the ray is not parallel to the plane of the triangle. White pseudocode for a routine that (1) determines whether the ray intersects the triangle and if so (2) finds the intersection point. Note that you are not merely intersecting the ray with the plane of the triangle. (Unlike the ray-plane calculation from the homework, both sides of the triangle are considered to be outside.) You may use matrix and vector operations (including more complicated ones like matrix inverse, matrix determinant, or solving a linear system) in your pseudocode. Hint: you learned how to test for a point inside a triangle in project 1.

Let P be the intersection point. Then, $P = \vec{u}+t\vec{v}$ with $t \geq 0$. Since P is in the triangle, $P = \alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$, with $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. Let $\alpha = 1 - \beta - \gamma$, $\vec{x} = \vec{b} - \vec{a}$, $\vec{y} = \vec{c} - \vec{a}$, and $\vec{z} = \vec{u} - \vec{a}$.

$$\begin{aligned} P &= \vec{u} + t\vec{v} \\ P &= \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \\ \vec{u} + t\vec{v} &= (1 - \beta - \gamma)\vec{a} + \beta\vec{b} + \gamma\vec{c} \\ \vec{z} + t\vec{v} &= \beta\vec{x} + \gamma\vec{y} \\ \underbrace{\begin{pmatrix} -\vec{v} & \vec{x} & \vec{y} \end{pmatrix}}_{\text{matrix}} \begin{pmatrix} t \\ \beta \\ \gamma \end{pmatrix} &= \vec{z} \end{aligned}$$

Thus, the intersection routine would do the following:

- Compute $\vec{x} = \vec{b} - \vec{a}$, $\vec{y} = \vec{c} - \vec{a}$, and $\vec{z} = \vec{u} - \vec{a}$.
- Compute t, β, γ by solving the linear system

$$\begin{pmatrix} -\vec{v} & \vec{x} & \vec{y} \end{pmatrix} \begin{pmatrix} t \\ \beta \\ \gamma \end{pmatrix} = \vec{z}$$

- Compute $\alpha = 1 - \beta - \gamma$
- Return no intersection if $t < 0$ or $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$
- Otherwise, return the intersection location $P = \vec{u} + t\vec{v}$

Note that t, β, γ can be computed directly. For example, taking the dot product of both sides of the equation $\vec{z} + t\vec{v} = \beta\vec{x} + \gamma\vec{y}$ by $\vec{v} \times \vec{x}$ gives $\vec{z} \cdot (\vec{v} \times \vec{x}) = \gamma\vec{y} \cdot (\vec{v} \times \vec{x})$, which can be solved for γ.