CS 130, Homework 6

Solutions

The intersection of a ray and and object can be described as a list of pairs. Each pair (a, b) indicates that the ray enters the object at $t = a$ and leaves it at $t = b$. If the ray exits and enters again, there will be more than one pair in the list. If the starting point of the ray is inside, then the first pair will look like $(0, b)$, which indicates that the ray starts inside $(t = 0)$ and exits at $t = b$. If the object happens to be infinite and the ray remains inside the object after some time $t = a$, then the last pair will look like (a, ∞). If the entire ray is inside, then there is one pair $(0, \infty)$. Note that intersections that occur at $t < 0$ are ignored, since these are not in the line of sight.

On the previous homework, we worked through the logic for intersecting a ray with a sphere and a plane. We have also looked at Boolean operations, which allow us to deal with unions, intersections, and differences of objects. The second project includes one more primitive, a cylinder. The cylinder is more complex than a sphere or a plane, since it is composed of three pieces: a top, a bottom, and a curved piece. Part of the goal of this homework assignment is to guide you through the process of solving the ray-cylinder problem. Since the goal of this is to eventually write code, your answers should be in a form of equations or inequalities that you could implement in code. If a case cannot occur, you just need to point this out.

Problem 1

A cylinder for our purposes is defined by a radius r and two points p and q. The line connecting the two points is the rotational axis of the cylinder. p and q lie at the center of the top and bottom faces of the cylinder. For this problem, we will ignore the top and bottom. The cylinder is now a rod of radius r that extends forever in both directions. The central axis of the rod passes through p and q. We want to compute the intersection of a ray $f(t) = \mathbf{u} + t\mathbf{v}$ with this surface. Assume $\mathbf{p} \neq \mathbf{q}$.

(a) Given a point x, let d be the distance between the point x and the line passing through p and q. (Note that a line extends forever in both directions.) Express d in terms of p, q, and x. You can simplify your work by using $\mathbf{w} = \mathbf{q} - \mathbf{p}$ and $\hat{\mathbf{w}} = \frac{\mathbf{w}}{\|\mathbf{w}\|}$.

(b) Under what conditions is the point x inside the cylinder (count the boundary as inside)? This should be a mathematical condition on r, p, q, and x. Under what conditions is the point x on the boundary of the cylinder? (Tip: you should be able to use the identity $(\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2$ to simplify your result.)

(c) Find all values of t where an intersection between the ray and the *surface* of the infinite cylinder. Do not worry at this stage whether t is positive or negative. (Tip: use $\mathbf{y} = \mathbf{u} - \mathbf{p}$, $\overline{\mathbf{y}} = \mathbf{y} \times \hat{\mathbf{w}}$, and $\overline{\mathbf{v}} = \mathbf{v} \times \hat{\mathbf{w}}$ to simplify your work.)

(d) Under what conditions are there no intersections? Note that there is more than one way in which this can occur.

(e) In case (d), there are two possibilities: the ray is always inside or always outside the cylinder. Determine when each occurs. (Both are possible here.)

(f) Under what conditions are there exactly two intersections (t_1 and t_2, with $t_1 < t_2$)? (If there is exactly one intersection $t = a$, then consider this as two: one entering and one leaving at time $t = a$, corresponding to the pair (a, a).)

(g) Divide case (f) into three cases. Case I: Both t values are negative. Case II: Both t values

are nonnegative. Case III: One t value is negative and the other is nonnegative. Under what conditions does each case occur? (You may express your answer using t_1 and t_2, but simplify your result.)
(h) What pairs should be generated in Case I of (g)?
(i) What pairs should be generated in Case II of (g)?
(j) What pairs should be generated in Case III of (g)?
(k) Under what conditions are there more than two intersections (but only finitely many)? If possible, what pairs should be emitted and when?
(l) Under what conditions are there infinitely many intersections between the ray and the cylinder? If possible, what pairs should be emitted and when?
(m) Summarize your logic above in a table. Be sure that all possible cases occur in the table. Your table should list out all possible cases, the mathematical criteria under which the case occurs, and the pairs that should be emitted in that case. This will help you when you implement ray-cylinder intersections.

(a) There are many ways to do this. I will take a minimization approach. Choose a point $\mathbf{z} = \mathbf{p} + \hat{\mathbf{w}}t$ on the line through $\mathbf{p}$ and $\mathbf{q}$. The squared distance from $\mathbf{z}$ to $\mathbf{x}$ is

$$\begin{aligned} D &= \|\mathbf{p} + \hat{\mathbf{w}}t - \mathbf{x}\|^2 \\ &= (\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) - 2t(\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}} + t^2 \end{aligned}$$

The distance we want is the minimum of this (we get to vary t to make this as small as possible). This gives us

$$\begin{aligned} 0 = D' &= -2(\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}} + 2t \\ t &= (\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}} \\ d^2 = D_{\min} &= (\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) - 2t(\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}} + t^2 \\ &= (\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) - 2((\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}})((\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}}) + ((\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}})^2 \\ &= (\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) - ((\mathbf{x} - \mathbf{p}) \cdot \hat{\mathbf{w}})^2 \\ &= \|(\mathbf{x} - \mathbf{p}) \times \hat{\mathbf{w}}\|^2 \\ d &= \|(\mathbf{x} - \mathbf{p}) \times \hat{\mathbf{w}}\| \end{aligned}$$

The same result is obtained by projecting $\mathbf{x}$ onto the line.
(b)

$$\begin{aligned} r &\geq d \\ r &\geq \|(\mathbf{x} - \mathbf{p}) \times \hat{\mathbf{w}}\| \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{x} &= \mathbf{u} + t\mathbf{v} \\ r^2 &= ((\mathbf{u} + t\mathbf{v} - \mathbf{p}) \times \hat{\mathbf{w}}) \cdot ((\mathbf{u} + t\mathbf{v} - \mathbf{p}) \times \hat{\mathbf{w}}) \\ r^2 &= ((\mathbf{y} + t\mathbf{v}) \times \hat{\mathbf{w}}) \cdot ((\mathbf{y} + t\mathbf{v}) \times \hat{\mathbf{w}}) \qquad \mathbf{y} = \mathbf{u} - \mathbf{p} \\ r^2 &= (\mathbf{y} \times \hat{\mathbf{w}}) \cdot (\mathbf{y} \times \hat{\mathbf{w}}) + 2t(\mathbf{v} \times \hat{\mathbf{w}}) \cdot (\mathbf{y} \times \hat{\mathbf{w}}) + t^2(\mathbf{v} \times \hat{\mathbf{w}}) \cdot (\mathbf{v} \times \hat{\mathbf{w}}) \\ r^2 &= \overline{\mathbf{y}} \cdot \overline{\mathbf{y}} + 2t\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} + t^2\overline{\mathbf{v}} \cdot \overline{\mathbf{v}} \qquad \overline{\mathbf{y}} = \mathbf{y} \times \hat{\mathbf{w}}, \overline{\mathbf{v}} = \mathbf{v} \times \hat{\mathbf{w}}, \end{aligned}$$

$$
\begin{aligned}
t &= \frac{-2\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} \pm \sqrt{(2\overline{\mathbf{v}} \cdot \overline{\mathbf{y}})^2 - 4(\overline{\mathbf{y}} \cdot \overline{\mathbf{y}} - r^2)(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}})}}{2\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}} \\
&= \frac{-\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} \pm \sqrt{(\overline{\mathbf{v}} \cdot \overline{\mathbf{y}})^2 - (\overline{\mathbf{y}} \cdot \overline{\mathbf{y}} - r^2)(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}})}}{\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}} \\
&= \frac{-\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} \pm \sqrt{(\overline{\mathbf{v}} \cdot \overline{\mathbf{y}})^2 - (\overline{\mathbf{y}} \cdot \overline{\mathbf{y}})(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}) + r^2(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}})}}{\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}} \\
&= \frac{-\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} \pm \sqrt{r^2(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}) - \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|^2}}{\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}}
\end{aligned}
$$

(d-e) One thing that can go wrong is $\overline{\mathbf{v}} = \mathbf{0}$, in which case we divide by zero. The other is $r^2(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}) < \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|^2$ or $r\|\overline{\mathbf{v}}\| < \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|$.

The criterion $r\|\overline{\mathbf{v}}\| < \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|$ will prevent an intersection (note that it also implies $\overline{\mathbf{v}} \neq \mathbf{0}$, so that we do not divide by zero). This corresponds to no intersection. (No pairs produced.)

The divide by zero case completely messes up the formula we derived for t. Lets return to the derivation before we divide by zero with $\overline{\mathbf{v}} = \mathbf{0}$.

$$
\begin{aligned}
r^2 &= \overline{\mathbf{y}} \cdot \overline{\mathbf{y}} + 2t\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} + t^2\overline{\mathbf{v}} \cdot \overline{\mathbf{v}} \\
r^2 &= \overline{\mathbf{y}} \cdot \overline{\mathbf{y}}
\end{aligned}
$$

Thus, we have $r = \|\overline{\mathbf{y}}\|$. In particular, if this is true, the ray is entirely on the surface of the cylinder (true for all t). Otherwise, it never intersects the surface. We can figure out if we are always inside or never inside by testing the ray's endpoint $\mathbf{u}$. We are inside if $r \geq \|\overline{\mathbf{y}}\|$, and produce the pair $(0, \infty)$. Otherwise, we are outside and produce no pairs.

(f) We are in this case if $\|\overline{\mathbf{v}}\| > 0$ and $r\|\overline{\mathbf{v}}\| \geq \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|$, since then there are two distinct real roots for t.

$$
\begin{aligned}
t_1 &= \frac{-\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} - \sqrt{r^2(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}) - \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|^2}}{\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}} \\
t_2 &= \frac{-\overline{\mathbf{v}} \cdot \overline{\mathbf{y}} + \sqrt{r^2(\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}) - \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|^2}}{\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}}
\end{aligned}
$$

(g) Case I: $t_2 < 0$. Case II: $t_1 \geq 0$. Case III: $t_1 < 0 \leq t_2$.
(h) No pairs.
(i) (t_1, t_2).
(j) $(0, t_2)$.
(k) This cannot happen.
(l) We found above that this occurs when $\overline{\mathbf{v}} = \mathbf{0}$ and $r = \|\overline{\mathbf{y}}\|$.

(m)

$\overline{\mathbf{v}} = \mathbf{0}$	$r \geq \|\overline{\mathbf{y}}\|$	$r\|\overline{\mathbf{v}}\| < \|\overline{\mathbf{v}} \times \overline{\mathbf{y}}\|$	$t_2 < 0$	$t_1 < 0$	parts	output
T	T	-	-	-	(d,e,l)	$\{(0, \infty)\}$
T	F	-	-	-	(d,e)	$\{\}$
F	-	T	-	-	(d,e)	$\{\}$
F	-	F	T	-	(f,g-I,h)	$\{\}$
F	-	F	F	T	(f,g-III,j)	$\{0, t_2\}$
F	-	F	F	F	(d,g-II,i)	$\{t_1, t_2\}$

Problem 2

In this problem, we will use the results from the first problem to address a cylinder with a top and a bottom face. To do this, we will consider the cylinder to be an intersection of the infinite cylinder from Problem 1 and two planes (half-spaces).

(a) What is the (outward) normal direction for the face of the cylinder passing through p?

Use this to construct an inequality describing which points x are inside. The plane itself, as well as the the side of the plane containing the cylinder, should be inside.
(b) Repeat (a) for the face containing point q. How are the normal directions for (a) and (b) related?
(c) We worked out the logic for which pairs should be produced by the intersections with the planes in parts (a) and (b) on the last homework. You may use those results here. Make a table listing the possible cases for intersections with (a) on the left and the possible cases for (b) on the top. List both the mathematical criterion and the pairs produced. (The table in the solutions posted for last homework summarize this in 6 cases, but these can be combined into 4 cases. If I do this, the table I draw here will be 4×4, and there will be 16 cases. If you use a different number of cases, your table will have a different number of entries.)
(d) For the table you produced in (c), identify the cases that are impossible. Place an "X" in these table entries. You should be able to determine which are not possible by noting that in those cases it is not possible to satisfy the criteria for both planes.
(e) For each of the remaining table entries, determine the intersection of the intersection pairs. (Each table entry that does not have an "X" gets a list of pairs.) Note that many of the cases are symmetrical. That makes things easier.
(f) For each of the remaining table entries, combine the criteria corresponding to the row and the column to get the criteria for the table entry. Simplify them. Again, symmetry makes this easier.
(g) Summarize your results in a table.

(a) Let $\mathbf{n} = \mathbf{n_p} = \frac{\mathbf{p}-\mathbf{q}}{\|\mathbf{p}-\mathbf{q}\|}$. This is the normal direction for the plane at $\mathbf{p}$. Inside is given by $(\mathbf{x}-\mathbf{p}) \cdot \mathbf{n} \le 0$.
(b) $\mathbf{n_q} = -\mathbf{n}$. Inside is given by $(\mathbf{x}-\mathbf{q}) \cdot \mathbf{n_q} \le 0$ or $(\mathbf{x}-\mathbf{q}) \cdot \mathbf{n} \ge 0$.
(c) The table from the solutions of hw 5 is:

$(\mathbf{u}-\mathbf{p}) \cdot \mathbf{n} \le 0$	$\mathbf{v} \cdot \mathbf{n}$	parts	output
T	< 0	(d,e)	$\{(0,\infty)\}$
T	> 0	(d,f)	$\{(0,t)\}$
T	$= 0$	(b,c)	$\{(0,\infty)\}$
F	< 0	(d,g)	$\{(t,\infty)\}$
F	> 0	(d,e)	$\{\}$
F	$= 0$	(b)	$\{\}$

This can be summarized in 4 cases by combining ones that produce the same output.

A: $(\mathbf{u}-\mathbf{p}) \cdot \mathbf{n} \le 0$ and $\mathbf{v} \cdot \mathbf{n} \le 0$; output $\{(0,\infty)\}$

B: $(\mathbf{u}-\mathbf{p}) \cdot \mathbf{n} \le 0$ and $\mathbf{v} \cdot \mathbf{n} > 0$; output $\{(0,t_p)\}$

C: $(\mathbf{u}-\mathbf{p}) \cdot \mathbf{n} > 0$ and $\mathbf{v} \cdot \mathbf{n} < 0$; output $\{(t_p,\infty)\}$

D: $(\mathbf{u}-\mathbf{p}) \cdot \mathbf{n} > 0$ and $\mathbf{v} \cdot \mathbf{n} \ge 0$; output $\{\}$

For the plane through $\mathbf{q}$, the conditions are similar. Note that the normal is negated, so the inequalities get reversed.

E: $(\mathbf{u}-\mathbf{q}) \cdot \mathbf{n} \ge 0$ and $\mathbf{v} \cdot \mathbf{n} \ge 0$; output $\{(0,\infty)\}$

F: $(\mathbf{u}-\mathbf{q}) \cdot \mathbf{n} \ge 0$ and $\mathbf{v} \cdot \mathbf{n} < 0$; output $\{(0,t_q)\}$

G: $(\mathbf{u}-\mathbf{q}) \cdot \mathbf{n} < 0$ and $\mathbf{v} \cdot \mathbf{n} > 0$; output $\{(t_q,\infty)\}$

H: $(\mathbf{u}-\mathbf{q}) \cdot \mathbf{n} < 0$ and $\mathbf{v} \cdot \mathbf{n} \le 0$; output $\{\}$

This is what the table looks like now. I will fill in entries as I go in a new table.

	E	F	G	H
A				
B				
C				
D				

(d) Note that conditions like $\mathbf{v}\cdot\mathbf{n} < 0$ and $\mathbf{v}\cdot\mathbf{n} > 0$ cannot both be true. This eliminates several combinations.

	E	F	G	H
A			✗	
B		✗		✗
C	✗		✗	
D		✗		

Another combination that is impossible is $a = (\mathbf{u}-\mathbf{p})\cdot\mathbf{n} > 0$ and $b = (\mathbf{u}-\mathbf{q})\cdot\mathbf{n} < 0$. To see this note that $a - b = (\mathbf{q}-\mathbf{p})\cdot\mathbf{n} = -\|\mathbf{q}-\mathbf{p}\|\mathbf{n}\cdot\mathbf{n} = -\|\mathbf{q}-\mathbf{p}\| \leq 0$. ($a > 0$ and $b < 0$ implies $a - b > 0$.) Including this in the table gives

	E	F	G	H
A			✗	
B		✗		✗
C	✗		✗	✗
D		✗	✗	✗

(e) The unique cases that are left are AE, AF, AH, and BG. (The others are symmetrical: $BE = AF$, $CF = BG$, and $DE = AH$; they are listed in the table in (g).)

- Case AE. $\{(0,\infty)\cap(0,\infty)\}$. Simplifies to: $\{(0,\infty)\}$.
- Case AF. $\{(0,\infty)\cap(0,t_q)\}$. Simplifies to: $\{(0,t_q)\}$.
- Case AH. $\{(0,\infty)\cap\emptyset\}$. Simplifies to: $\{\}$.
- Case BG. $\{(0,t_p)\cap(t_q,\infty)\}$. Simplifies to: $\{(t_q,t_p)\}$.

(f) The unique cases that are left are AE, AF, AH, and BG. (The others are symmetrical: $BE = AF$, $CF = BG$, and $DE = AH$; they are listed in the table in (g).)

- Case AE. $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $\mathbf{v}\cdot\mathbf{n} \leq 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} \geq 0$ and $\mathbf{v}\cdot\mathbf{n} \geq 0$. Simplifies to: $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} \geq 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$.
- Case AF. $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $\mathbf{v}\cdot\mathbf{n} \leq 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} \geq 0$ and $\mathbf{v}\cdot\mathbf{n} < 0$. Simplifies to: $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} \geq 0$ and $\mathbf{v}\cdot\mathbf{n} < 0$.
- Case AH. $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $\mathbf{v}\cdot\mathbf{n} \leq 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} < 0$ and $\mathbf{v}\cdot\mathbf{n} \leq 0$. Simplifies to: $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} < 0$ and $\mathbf{v}\cdot\mathbf{n} \leq 0$.
- Case BG. $(\mathbf{u}-\mathbf{p})\cdot\mathbf{n} \leq 0$ and $\mathbf{v}\cdot\mathbf{n} > 0$ and $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} < 0$ and $\mathbf{v}\cdot\mathbf{n} > 0$. Simplifies to: $(\mathbf{u}-\mathbf{q})\cdot\mathbf{n} < 0$ and $\mathbf{v}\cdot\mathbf{n} > 0$.

(g) The result is shown in the table below. The comparisons marked "-" are true, but they are implied by a "F" from the other comparison

$(\mathbf{u}-\mathbf{p})\cdot\mathbf{n}\le 0$	$(\mathbf{u}-\mathbf{q})\cdot\mathbf{n}\ge 0$	$\mathbf{v}\cdot\mathbf{n}$	case	output
T	T	$=0$	AE	$\{(0,\infty)\}$
T	T	>0	BE	$\{(0,t_p)\}$
T	T	<0	AF	$\{(0,t_q)\}$
-	F	≤ 0	AH	$\{\}$
-	F	>0	BG	$\{(t_q,t_p)\}$
F	-	≥ 0	DE	$\{\}$
F	-	<0	CF	$\{(t_p,t_q)\}$

Problem 3

To complete the intersection routine, we must intersect the results of Problems 1 and 2. We will do this now. If you have done problems 1 and 2 correctly, you should always get zero or one pair (a,b) from Problem 1 and zero or one pair (c,d) from Problem 2. (If they exist, $0\le a\le b$ and $0\le c\le d$.) You do not need to be concerned with what the pairs actually are at this stage. You may complete this stage even if you have not solved the previous problems.

(a) If no pairs were produced by Problem 1 or Problem 2, what pairs should be output? Be sure to handle all possible cases.
(b) If no pairs were produced by Problem 1 but Problem 2 produced (c,d), what pairs should be output? Be sure to handle all possible cases.
(c) If no pairs were produced by Problem 2 but Problem 1 produced (a,b), what pairs should be output? Be sure to handle all possible cases.
(d) If Problem 1 produced (a,b) and Problem 2 produced (c,d), what pairs should be output? Be sure to handle all possible cases.

(a-c) A ray must intersect both objects in order to intersect their Boolean intersection. Thus, there are no intersections in these cases, and no pairs should be produced.
(d) We need to compute the intersection of the intervals $I=[a,b]\cap[c,d]$. If $x\in I$, then $x\ge a$ and $x\ge c$. Thus, $x\ge m=\max(a,c)$. Similarly, $x\le n=\min(b,d)$. If $m>n$, then $I=\emptyset$, and no pairs are produced. Otherwise, $I=[m,n]$, and the pair (m,n) should be produced.