CS 130, Homework 5

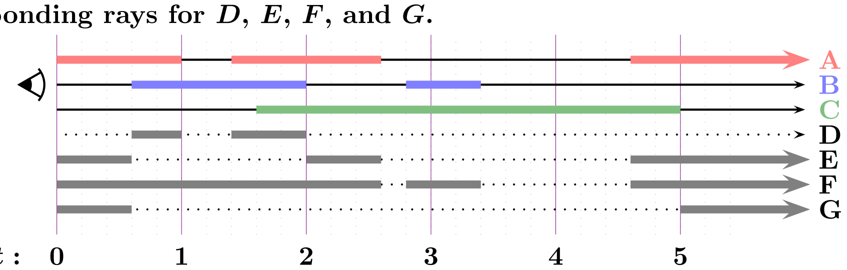
Solutions

The intersection of a ray and and object can be described as a list of pairs. Each pair (a, b) indicates that the ray enters the object at $t = a$ and leaves it at $t = b$. If the ray exits and enters again, there will be more than one pair in the list. If the starting point of the ray is inside, then the first pair will look like $(0, b)$, which indicates that the ray starts inside ($t = 0$) and exits at $t = b$. If the object happens to be infinite and the ray remains inside the object after some time $t = a$, then the last pair will look like (a, ∞). If the entire ray is inside, then there is one pair $(0, \infty)$. For example, in the first problem, $A = \{(0.0, 1.0), (1.4, 2.6), (4.6, \infty)\}$, $B = \{(0.6, 2.0), (2.8, 3.4)\}$, and $C = \{(1.6, 5.0)\}$. This is the representation that will be used for the second project. Note that intersections that occur at $t < 0$ are ignored, since these are not in the line of sight.

In code, we just have a `std::vector` of floats. This will always contain an even number of floats, unless the last entry is ∞. If the last entry is ∞, then it is omitted, resulting in an odd number of entries. The floats in the `std::vector` should always be nonnegative and in increasing order.

Problem 1

Composite objects are made from objects A, B, and C. Below are rays depicting what portions of the ray are inside each object. Let $D = A \cap B$, $E = A - B$, $F = A \cup B$, and $G = A - (B \cup C)$. Sketch corresponding rays for D, E, F, and G.



Problem 2

Let $f(t) = \mathbf{u} + t\mathbf{v}$ define a ray ($t \geq 0$; $\|\mathbf{v}\| = 1$). A half-space is defined by $(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} \leq 0$ ($\|\mathbf{n}\| = 1$). Note that this is really a plane, where we consider one side of the plane to be inside and the other side to be outside. The normal vector points outside. Treating planes like this is more useful for our purposes than defining a plane as a thin object.

(a) Find all values of t where an intersection between the ray and the *surface* of the half-space (plane) occurs. Don't worry about whether t is positive or negative yet.
(b) Under what conditions are there no such intersections?
(c) When should (b) be emitted as $(0, \infty)$, since the ray starts inside the half-space and never leaves it?
(d) Under what conditions is there exactly one such intersection?
(e) When should the intersection from (d) be discarded since all intersection between the half-space and ray occurs when $t < 0$? When this occurs, identify when the pair $(0, \infty)$ should be emitted.
(f) When should the intersection from (d) be emitted as $(0, b)$, since the ray starts inside the half-space but then leaves it?

(g) When should the intersection from (d) be emitted as (a, ∞), since the ray starts outside, enters the half-space but then never leaves it?
(h) Under what conditions are there two or more intersection with the *surface* of the half-space? Repeat the steps (e)-(g) for these pairs.
(i) Summarize your logic above in a table. Be sure that all possible cases occur in the table. This will help you when you implement ray-plane intersections.

(a)

$$\begin{aligned}
\mathbf{x} &= \mathbf{u} + t\mathbf{v} && \text{(ray)} \\
(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} &= 0 && \text{(plane)} \\
(\mathbf{u} + t\mathbf{v} - \mathbf{p}) \cdot \mathbf{n} &= 0 && \text{(intersection)} \\
(\mathbf{u} - \mathbf{p}) \cdot \mathbf{n} + t(\mathbf{v} \cdot \mathbf{n}) &= 0 \\
t &= -\frac{(\mathbf{u} - \mathbf{p}) \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}
\end{aligned}$$

(h) This can never happen.
(b-g,i) The rest of the cases depend on whether the ray starts inside ($(\mathbf{u} - \mathbf{p}) \cdot \mathbf{n} \leq 0$), whether the ray is heading in the normal direction ($\mathbf{v} \cdot \mathbf{n} > 0$), or parallel to the surface plane ($\mathbf{v} \cdot \mathbf{n} = 0$). These are best summarized in a table.

$(\mathbf{u} - \mathbf{p}) \cdot \mathbf{n} \leq 0$	$\mathbf{v} \cdot \mathbf{n}$	parts	output
T	< 0	(d,e)	$\{(0, \infty)\}$
T	> 0	(d,f)	$\{(0, t)\}$
T	$= 0$	(b,c)	$\{(0, \infty)\}$
F	< 0	(d,g)	$\{(t, \infty)\}$
F	> 0	(d,e)	$\{\}$
F	$= 0$	(b)	$\{\}$

Problem 3

Let $f(t) = \mathbf{u} + t\mathbf{v}$ define a ray ($t \geq 0$; $\|\mathbf{v}\| = 1$). A sphere has radius r and center $\mathbf{c}$.
(a) Find all values of t where an intersection between the ray and the surface of the sphere occurs. Don't worry about whether t is positive or negative yet.
(b) Under what conditions are there no intersections?
(c) Under what conditions are there exactly two intersections? (If there is exactly one intersection $t = a$, then consider this as two: one entering and one leaving at time $t = a$, corresponding to the pair (a, a).)
(d) When should the pair from (c) be discarded since both intersections are negative?
(e) When should the pair from (c) be replaced by $(0, b)$, since one intersections is negative?
(f) When will the pair from (c) be of the form (a, ∞)?
(g) Under what conditions are there more than two intersections? Repeat the steps (d)-(f) for these pairs.
(h) Summarize your logic above in a table. Be sure that all possible cases occur in the table. This will help you when you implement ray-sphere intersections.

(a)

$$\begin{aligned}
\mathbf{x} &= \mathbf{u} + t\mathbf{v} && \text{(ray)}\\
\|\mathbf{x}-\mathbf{c}\|^2 &= r^2 && \text{(sphere)}\\
\|\mathbf{u}+t\mathbf{v}-\mathbf{c}\|^2 &= r^2 && \text{(intersection)}\\
(\mathbf{u}-\mathbf{c})\cdot(\mathbf{u}-\mathbf{c}) + 2(\mathbf{u}-\mathbf{c})\cdot(t\mathbf{v}) + (t\mathbf{v})\cdot(t\mathbf{v}) &= r^2\\
t^2 + (2(\mathbf{u}-\mathbf{c})\cdot\mathbf{v})t + (\mathbf{u}-\mathbf{c})\cdot(\mathbf{u}-\mathbf{c}) - r^2 = 0 \qquad (\mathbf{v}\cdot\mathbf{v}=1)t^2 + (2\mathbf{w}\cdot\mathbf{v})t + \mathbf{w}\cdot\mathbf{w} - r^2 &= 0 && (\mathbf{w}=\mathbf{u}-\mathbf{c})
\end{aligned}$$

$$\begin{aligned}
t &= \frac{-2\mathbf{w}\cdot\mathbf{v} \pm \sqrt{(2\mathbf{w}\cdot\mathbf{v})^2 - 4(\mathbf{w}\cdot\mathbf{w}-r^2)}}{2}\\
t &= -\mathbf{w}\cdot\mathbf{v} \pm \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
t_0 &= -\mathbf{w}\cdot\mathbf{v} - \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
t_1 &= -\mathbf{w}\cdot\mathbf{v} + \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}
\end{aligned}$$

(b) No real solution when: $(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2 < 0$
(c) Two (or one) solutions: $(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2 \geq 0$. Note that $t_1 \geq t_0$.
(d) The pair should be discarded when

$$\begin{aligned}
t_1 &= -\mathbf{w}\cdot\mathbf{v} + \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
0 &> t_1\\
0 &> -\mathbf{w}\cdot\mathbf{v} + \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
\mathbf{w}\cdot\mathbf{v} &> \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2} && \text{(imples } \mathbf{w}\cdot\mathbf{v} > 0)\\
(\mathbf{w}\cdot\mathbf{v})^2 &> (\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2\\
\mathbf{w}\cdot\mathbf{w} &> r^2
\end{aligned}$$

This occurs when $\mathbf{w}\cdot\mathbf{v} > 0$ and $\mathbf{w}\cdot\mathbf{w} > r^2$.
(e) The pair should be replaced with $(0, t_1)$ when $t_0 \leq 0 \leq t_1$.

$$\begin{aligned}
t_0 &= -\mathbf{w}\cdot\mathbf{v} - \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
0 &\geq t_0\\
0 &\geq -\mathbf{w}\cdot\mathbf{v} - \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2} && \text{(true if } \mathbf{w}\cdot\mathbf{v} > 0\text{; assume } \mathbf{w}\cdot\mathbf{v} \leq 0)\\
\mathbf{w}\cdot\mathbf{v} &\geq -\sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
-\mathbf{w}\cdot\mathbf{v} &\leq \sqrt{(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2}\\
(\mathbf{w}\cdot\mathbf{v})^2 &\leq (\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2\\
\mathbf{w}\cdot\mathbf{w} &\leq r^2
\end{aligned}$$

We need: ($\mathbf{w}\cdot\mathbf{v} > 0$ and $\mathbf{w}\cdot\mathbf{w} \leq r^2$) or ($\mathbf{w}\cdot\mathbf{v} \leq 0$ and $\mathbf{w}\cdot\mathbf{w} \leq r^2$) or simply $\mathbf{w}\cdot\mathbf{w} \leq r^2$. Note that in the case $\mathbf{w}\cdot\mathbf{v} > 0$ we needed to be careful to exclude case (d).
(f) The sphere is finite. This cannot happen.
(g) No more than two intersections are possible. This cannot happen.
(h) The cases are summarized as

$(\mathbf{w}\cdot\mathbf{v})^2 - \mathbf{w}\cdot\mathbf{w} + r^2 < 0$	$\mathbf{w}\cdot\mathbf{v} > 0$	$\mathbf{w}\cdot\mathbf{w} > r^2$	output
T	-	-	$\{\}$
F	T	T	$\{\}$
F	T	F	$\{0, t_1\}$
F	F	F	$\{0, t_1\}$
F	F	T	$\{t_0, t_1\}$

Problem 4

Let $x = 2i + 3j$, $y = 2 + 3i - k$, and $z = -1 + j + k$ be three quaternions. Compute each of the following.

(a) $y + z$
(b) $y - z$
(c) yz
(d) zy
(e) $\overline{z}$
(f) z^{-1}
(g) zxz^{-1}
(h) $|z|$
(i) z^2
(j) k^2
(k) jk

Let $u = \langle 2, 3, 0\rangle$, $v = \langle 3, 0, -1\rangle$, and $w = \langle 0, 1, 1\rangle$. We will need: $v \cdot w = -1$, $v \times w = \langle 1, -3, 3\rangle$.
(a) $1 + 3i + j$
(b) $3 + 3i - j - 2k$
(c) $((2)(-1) - v \cdot w, (-1)v + 2w + v \times w) = (-1, \langle -2, -1, 6\rangle) = -1 - 2i - j + 6k$
(d) $((2)(-1) - v \cdot w, (-1)v + 2w + w \times v) = (-1, \langle -4, 5, 0\rangle) = -1 - 4i + 5j$
(e) $-1 - j - k$
(f) $-\frac{1}{3} - \frac{1}{3}j - \frac{1}{3}k$
(g) $\frac{4}{3}i - \frac{1}{3}j + \frac{10}{3}k$
(h) $\sqrt{3}$
(i) $-1 - 2j - 2k$
(j) -1
(k) i