CS 130, Homework 1

Solutions

Please complete the problems below. Be sure to show your work; answers alone are not enough.

Problem 1

Using the definitions below, compute the requested quantities. If the quantity does not exist, write "DNE" and give a very brief explanation.

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$
(b) $\mathbf{A}^T\mathbf{A} - \mathbf{B}$
(c) $\mathbf{A}\mathbf{A}^T - \mathbf{B}$
(d) A vector of unit length that is orthogonal to both u and v
(e) A vector of the form $\alpha\mathbf{u} + \beta\mathbf{v}$ which is orthogonal to v. (α, β are scalars.)
(f) Two vectors w and x such that $\mathbf{w} + \mathbf{x} = \mathbf{u}$, w is parallel to v, and x is orthogonal to v.

(a)

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u}}{\sqrt{1^2 + (-2)^2 + 0^2}} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$$

(b)

$$\begin{aligned}
\mathbf{A}^T\mathbf{A} - \mathbf{B} &= \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (1)(1) + (-1)(-1) + (2)(2) & (1)(1) + (-1)(0) + (2)(3) \\ (1)(1) + (0)(-1) + (3)(2) & (1)(1) + (0)(0) + (3)(3) \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 7 \\ 7 & 10 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 7 \\ 6 & 9 \end{pmatrix}
\end{aligned}$$

(c) Does not exist. $\mathbf{A}\mathbf{A}^T$ is 3×3, while $\mathbf{B}$ is 2×2.
(d) The vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, though it is not of unit length. This can be

corrected by normalizing it. Thus, $\frac{w}{\|w\|}$ is a solution. There are two solutions; $-\frac{w}{\|w\|}$ is the other.

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2)(1) - (0)(1) \\ (0)(3) - (1)(1) \\ (1)(1) - (-2)(3) \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 7 \end{pmatrix}$$

$$\|\mathbf{w}\| = \sqrt{(-2)^2 + (-1)^2 + 7^2} = \sqrt{54}$$

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \begin{pmatrix} -\frac{2}{\sqrt{54}} \\ -\frac{1}{\sqrt{54}} \\ \frac{7}{\sqrt{54}} \end{pmatrix}$$

(e) Let $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$, so that $\mathbf{w} \cdot \mathbf{v} = 0$.

$$\begin{aligned} 0 = \mathbf{w} \cdot \mathbf{v} &= (\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{v} \\ &= \alpha\mathbf{u} \cdot \mathbf{v} + \beta\mathbf{v} \cdot \mathbf{v} \\ &= \alpha(3 - 2 + 0) + \beta(9 + 1 + 1) \\ &= \alpha + \beta 11 \end{aligned}$$

Let $\beta = 1$. Then, $\alpha = -11$. Finally,

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} = -11\mathbf{u} + \mathbf{v} = \begin{pmatrix} -11 + 3 \\ 22 + 1 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 23 \\ 1 \end{pmatrix}$$

(f) To be parallel, we need $\mathbf{w} = \alpha\mathbf{v}$ for some α. Then, $\mathbf{x} = \mathbf{u} - \alpha\mathbf{v}$. For orthogonality, $\mathbf{x} \cdot \mathbf{v} = (\mathbf{u} - \alpha\mathbf{v}) \cdot \mathbf{v} = (\mathbf{u} \cdot \mathbf{v}) - \alpha(\mathbf{v} \cdot \mathbf{v}) = 1 - 11\alpha$. Thus, $\alpha = \frac{1}{11}$. Then $\mathbf{w} = \alpha\mathbf{v} = \begin{pmatrix} \frac{3}{11} \\ \frac{1}{11} \\ \frac{1}{11} \end{pmatrix}$ and $\mathbf{x} = \mathbf{u} - \mathbf{w} = \begin{pmatrix} \frac{8}{11} \\ -\frac{23}{11} \\ -\frac{1}{11} \end{pmatrix}$.