Proof (cont’d)

Reduction $3SAT$ to $VERTEX-COVER$.

Let $\phi$ be an instance of $3SAT$ with $n$ variables and $m$ clauses. Define the graph $G$ as follows:

- **the nodes**: the literals $v_i, \overline{v_i}, 1 \leq i \leq n$, and
  - their occurrences $a_{i1}, a_{i2}, a_{i3} : 1 \leq i \leq m$,
  - a total of $3m + 2n$ nodes
- **the edges**: $(v_i, \overline{v_i}), 1 \leq i \leq n$;
  - $(a_{i1}, a_{i2}), (a_{i2}, a_{i3}), (a_{i3}, a_{i1}), 1 \leq i \leq m$;
  - for each $i, 1 \leq i \leq n$, and $j, 1 \leq j \leq 3$, connect $a_{ij}$ and its corresponding literal.

Proof (cont’d)

We claim that $G$ has an $(n + 2m)$ node vertex cover if and only if $\phi$ is in $3SAT$.

There are precisely $n$ edges of the type $(v_i, \overline{v_i}), 1 \leq i \leq n$. So an $n$-node vertex cover has to have at least one out of $v_i$ and $\overline{v_i}$ for every $i$.

There are precisely $m$ triangles for the clauses. At least two nodes have to be selected from each triangle to cover the triangle edges. So $2m$ nodes are needed.

More NP-Completeness

We know: $3SAT \leq_p CLIQUE$, $CLIQUE \in NP$, and $3SAT$ is NP-complete. So, $CLIQUE$ is NP-complete.

**Vertex Cover**

A vertex cover of an undirected graph is a subset of nodes such that every edge touches a member of the subset.

$VERTEX-COVER = \{ \langle G, k \rangle \mid G \text{ has a vertex cover of size } k \}$.

**Theorem.** $VERTEX-COVER$ is NP-complete.

**Proof** Proving $VERTEX-COVER \in NP$ is easy. Guess a bit for each node to decide whether or not select the node. Check whether precisely $k$ nodes are selected, if so, check whether the set of $k$ nodes is a vertex cover.
Proof (cont’d)

Let $\phi$ be a formula of $n$ variables and $m$ clauses. Introduce decimal numbers $y_1, \ldots, y_n, z_1, \ldots, z_m, c_1, \ldots, c_m, d_1, \ldots, d_m$, each of at most $n + m$ digits.

$y_i$: $y_i$ has a 1 at the $(m+i)$th digit and has a 1 at position $j$ if $x_i$ appears in the $j$th clause; all the other positions have a 0

$z_i$: $z_i$ has a 1 at the $(m+i)$th digit and has a 1 at position $j$ if $\overline{x_i}$ appears in the $j$th clause; all the other positions have a 0

$c_i, d_i$: $c_i$ has a 1 only at the $i$th position, $d_i$ has a 1 only at the $i$th position,

$S$: $S$ is the number that has a 3 at every position between 1 and $m$ and has a 1 at every position between $m + 1$ and $m + n$.

Subset-Sum is NP-complete

$SUBSET\text{-SUM}$ is the problem of, given a multiset of numbers $z_1, \ldots, z_m$ and a number $S$, whether there is subset $y_1, \ldots, y_k$ of $z_i$‘s such that $y_1 + \cdots + y_k = S$.

Theorem. $SUBSET\text{-SUM}$ is NP-complete.

Proof Reduce $3SAT$ to $SUBSET\text{-SUM}$. The construction is reminiscent of the reduction from $3SAT$ to $VERTEX\text{-COVER}$, where the reduction generates a graph whose $n + 2m$ node cover has a property that at least one “literal-occurrence” edge of each triangle is touched and the rest of the nodes in each triangle is touched.