CS 141, Winter 2018

Problem 1. (25 points) Solve the following recurrences:

1. \( T(n) = 2T(n/8) + n\log n \)
2. \( T(n) = 2T(n/4) + \sqrt{n} \)
3. \( T(n) = 9T(n/3) + 8n/3 \)
4. \( T(n) = 4T(n/2) + n^2\log n \)
5. \( T(n) = T(n/3) + T(2n/3) + n \)

Problem 2. (25 points) Suppose you have \( k \) sorted arrays, each with \( n \) elements, and you want to combine them into a single sorted array of \( kn \) elements. Describe a divide and conquer algorithm that takes \( O(kn \log k) \) time. Explain carefully why your algorithm takes \( O(kn \log k) \) time.

Problem 3. (25 points)

For an \( n \) that is a power of 2, the \( n \times n \) Weirdo matrix \( W_n \) is defined as follows. For \( n = 1 \), \( W_1 = [1] \). For \( n > 1 \), \( W_n \) is defined inductively by

\[
W_n = \begin{bmatrix}
W_{n/2} & -W_{n/2} \\
I_{n/2} & W_{n/2}
\end{bmatrix},
\]

where \( I_k \) denotes the \( k \times k \) identity matrix. For example,

\[
W_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad W_4 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad W_8 = \begin{bmatrix}
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 0 & -1 & 1 \\
0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

Give \( O(n \log n) \)-time algorithm that computes the product \( W_n \cdot \bar{x} \), where \( \bar{x} \) is a vector of length \( n \) and \( n \) is a power of 2.

Problem 4. (25 points) Given an array of numbers \( X = \{x_1, x_2, \ldots, x_n\} \), an exchanged pair in \( X \) is a pair \( (x_i, x_j) \) such that \( i < j \) and \( x_i > x_j \). Note that an element \( x_i \) can be part of up to \( n - 1 \) exchanged pairs, and that the maximal possible number of exchanged pairs in \( X \) is \( n(n - 1)/2 \), which is achieved if the array is sorted in descending order. Give a divide-and-conquer algorithm that counts the number of exchanged pairs in \( X \) in \( O(n \log n) \) time.