Problem 1. (20 points)

In the algorithm SELECT described in class (linear time selection), the input elements are divided in groups of 5. Write the recurrence relation for the time complexity of SELECT if you decided to divide the input in groups of 7. Would the new algorithm still work in linear time? Repeat the analysis if groups of 3 is used.

Problem 2. (40 points)

Assume that you are given an unsorted array $A$ containing $n$ distinct integers and another integer $k$. Your task is to reorder and divide $A$ into $k$ equal-sized groups, such that the integers in the first group are lower than the integers in the second group, and the integers in the second group are lower than the integers in the third group, and so on (however, the integers inside each group do not need to be sorted). Formally, you have to reorder and partition $A$ into groups $A_1, \ldots, A_k$ such that for all $1 \leq i < j \leq k$ and for all $a \in A_i$ and $b \in A_j$, we have $a < b$. Your algorithm should run in time $O(n \log k)$. You may assume that is $n$ divisible by $k$ and that $k$ is a power of two.
Problem 3. (40 points)

A $N \times N$ matrix is called Toeplitz if it has constant entries down its diagonals. For example

$$A = \begin{bmatrix}
a_0 & a_1 & a_2 & a_3 & \ldots & a_{N-1} \\
a_{-1} & a_0 & a_1 & a_2 & \ldots & \vdots \\
a_{-2} & a_{-1} & a_0 & a_1 & \ldots & a_3 \\
a_{-3} & a_{-2} & a_{-1} & a_0 & \ldots & a_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & a_1 \\
a_{-(N-1)} & \ldots & a_{-3} & a_{-2} & a_{-1} & a_0
\end{bmatrix}$$

Toeplitz matrices occur surprisingly often in several application domains (e.g. time-series analysis, the numerical solution of certain partial differential equations, approximation of functions, etc.).

1. Is the sum of two Toeplitz matrices necessarily Toeplitz? What about the product?

2. Describe how to represent a Toeplitz matrix so that two $n \times n$ Toeplitz matrices can be added in $O(n)$ time.

3. Give a $O(n \log n)$-time algorithm for multiplying an $n \times n$ Toeplitz matrix by a vector of length $n$. 