Weighted Graphs

CS 141, Spring 2019

Outline

• (single-source) shortest path
  – Dijkstra (Section 4.4)
  – Bellman-Ford (Section 4.6)
  – DAGs (Section 4.7)

• (all-pairs) shortest path
  – Floyd-Warshall (Section 6.6)

• minimum spanning tree
  – Kruskal (Section 5.1.3)
  – Prim (Section 5.1.5)
Shortest Path

- Let $G$ be a weighted graph ($w(e)$ is the weight of the edge $e$)

- The length of a path $P$ is the sum of the weights of the edges of $P$

- If $P=e_0,e_1,\ldots,e_{k-1}$ then the length of $P$ is $\sum w(e_i)$

Single-Source Shortest Path

- The distance from a vertex $u$ to vertex $v$, denoted by $d(u,v)$ is the length of a minimum length path (also called shortest-path) from $u$ to $v$, if such a path exists
- If the path does not exists, $d(u,v)=+\infty$
- Note that if there is a negative cycle, then the distance may not be defined
Optimal Substructure

- **Fact:** subpaths of shortest paths are shortest paths
- **Proof:** decompose a shortest path 
  \[ p = <v_1, v_2, ..., v_k> \] into \( v_i \rightarrow v_i \rightarrow v_j \rightarrow v_k \). Then 
  \[ w(p) = w(v_1, v_i) + w(v_i, v_j) + w(v_j, v_k) \]. If \( v_i \rightarrow v_j \) 
  is not optimal, then we could make the path 
  \( v_i \rightarrow v_k \) shorter, which contradicts the 
  optimality of \( p \).

Shortest-Path Problems

- **Single-source (single-destination).** Find a shortest path 
  from a given source (vertex \( s \)) to all the other vertices 
  positive weights \( \rightarrow \) greedy algorithm 
  pos. & neg. weights \( \rightarrow \) dynamic programming 
- **All-pairs.** Find shortest-paths for every pair of vertices 
  pos. & neg. weights \( \rightarrow \) dynamic programming 
- **Special cases** 
  - **Unweighted shortest-paths.** Use BFS 
  - **Shortest path on a DAG.** Next
Shortest path on a directed acyclic graphs (DAGs)

What is a directed graph?

- A graph $G=(V,E)$ is called *directed* when all its edges are directed.
- Each edge goes in one direction: edge $(a,b)$ goes from $a$ to $b$, but not $b$ to $a$.
Reachability and cycles

• Given vertices $u$ and $v$ of a digraph $G$, we say that $u$ reaches $v$ (and $v$ is reachable from $u$) if $G$ has a directed path from $u$ to $v$

• $G$ is strongly connected if, for any two vertices $u$ and $v$, $u$ reaches $v$ and $v$ reaches $u$

• A directed cycle of $G$ is a cycle where all edges in the cycle are traversed according to their respective orientation

• A digraph is acyclic if has not directed cycles

Problems on directed graphs

• Given $u$ and $v$ determine whether $u$ reaches $v$

• Find all vertices of $G$ that are reachable from a given vertex $s$

• Determine whether $G$ is strongly connected

• Determine whether $G$ is acyclic

• Given a directed acyclic graph $G$, find an ordering of the nodes that respects $G \Rightarrow$ topogical sorting

• Weighted: Shortest path
Shortest path on DAGs

- When the graph is acyclic (DAG), we can solve the single-source shortest path faster than using Dijkstra’s or Bellman-Ford’s.

- **Steps**
  - Run topological sorting (requires directed DFS)
  - Relax edges in the sorted order

Depth-First Search

- A directed DFS partition the edges of $G$ into 
  tree edges, and non-tree edges
- The non-tree edges can be divided in
  - **Back edges**: which connect a vertex to an ancestor in the DFS tree
  - **Forward edges**: which connect a vertex to a descendent in the DFS tree
  - **Cross edges**: which connect a vertex to a vertex that is neither its ancestor nor its descendent
Revised directed DFS Code

**Algorithm** Directed_DFS(u)

- u.color $\rightarrow$ YELLOW;
- time $\rightarrow$ time + 1;
- u.discovery $\rightarrow$ time;
- for each v in adjacency(u)
  - if (v.color = WHITE)
    - Directed_DFS(v);
- u.color $\rightarrow$ BLACK;
- time $\rightarrow$ time + 1;
- u.finish $\rightarrow$ time;

Directed DFS Example
Directed DFS Example

D   F
1     |
\   |
   \|
   \|
   d   f

Directed DFS Example

D   F
1     |
\   |
   \|
   \|
   d   f

D   F
2     |
\   |
   \|
   \|
   d   f
Directed DFS Example

Back edges: yellow to yellow
Directed DFS Example

Back edges: yellow to yellow

Directed DFS Example

Back edges: yellow to yellow
Directed DFS Example

Back edges: yellow to yellow
Cross edges: u yellow to v black and d[u] > d[v]

Back edges: yellow to yellow
Cross edges: u yellow to v black and d[u] > d[v]
Directed DFS Example

Back edges: yellow to yellow
Cross edges: u yellow to v black and d[u]>d[v]
Forward edges: u yellow to v black and d[u]<d[v]
Directed DFS Example

Back edges: yellow to yellow
Cross edges: u yellow to v black and \( d[u] > d[v] \)
Forward edges: u yellow to v black and \( d[u] < d[v] \)
Directed DFS Example

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Cross edges: u yellow to v black and d[u]>d[v]
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Directed DFS Example

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Forward edges: u yellow to v black and \( d[u] < d[v] \)
Directed DFS Example

Back edges: yellow to yellow
Cross edges: \( u \) yellow to \( v \) black and \( d[u] > d[v] \)
Forward edges: \( u \) yellow to \( v \) black and \( d[u] < d[v] \)
Directed DFS Example

Back edges: yellow to yellow
Cross edges: u yellow to v black and $d[u] > d[v]$  
Forward edges: u yellow to v black and $d[u] < d[v]$
Directed DFS Example

Back edges: yellow to yellow
Cross edges: u yellow to v black and d[u]>d[v]
Forward edges: u yellow to v black and d[u]<d[v]
Topological ordering

- A topological ordering of $G$ is an ordering $\{v_1, v_2, ..., v_n\}$ of the vertices of $G$ such that for every edge $(v_i, v_j), i < j$

- Topological ordering may not be unique

Figure 6.14: Two topological orderings of the same acyclic digraph.
Topological sorting

- Labels are increasing along a directed path

- **Theorem:** A digraph has a topological sorting if and only if it is acyclic (i.e., a DAG)

Topological sorting using directed DFS

`Topological-Sort(G)`

1. Call DFS$(G)$ from any node and compute the finish time $f[v]$ for each vertex $v$
2. As each vertex is finished, insert it onto the front of a linked list
3. Return the linked list of vertices

Time complexity $O(n+m)$
Example (using directed DFS)

Shortest path on DAGs

- When the graph is acyclic (DAG), we can solve the single-source shortest path faster than using Dijkstra’s or Bellman-Ford’s

- Steps
  - Run topological sorting (requires directed DFS)
  - Relax edges in the sorted order
Shortest path on DAGs

![Graph 1](image1)

![Graph 2](image2)

Shortest path on DAGs

![Graph 3](image3)

![Graph 4](image4)
Shortest path on DAGs

Algorithm DAGShortestPaths(G, s):
Input: A weighted directed acyclic graph (dag) \( G \) with \( n \) vertices and \( m \) edges, and a distinguished vertex \( s \) in \( G \)
Output: Vertex labels \( D \) such that \( D[v] \) is the distance from \( s \) to \( v \) in \( G \)
Compute a topological ordering \( (v_1, v_2, \ldots, v_n) \) for \( G \)
\( D[s] \leftarrow 0 \)
for each vertex \( v \) in \( G \) other than \( s \) do
\( D[v] \leftarrow +\infty \)
for \( i \leftarrow 1 \) to \( n - 1 \) do
{Relax each outgoing edge from \( v_i \)}
for each edge \( (v_i, u) \) do
if \( D[v_i] + w(v_i, u) \) < \( D[u] \) then
\( D[u] \leftarrow D[v_i] + w(v_i, u) \)
Output the distance labels \( D \) as the distances from \( s \).
Shortest path on DAGs

- Topological sorting runs in $O(n+m)$

- The algorithms check each node once, and each outgoing edge once

- Total time is $O(n+m)$

Dijkstra’s algorithm
Dijkstra’s algorithm

- Dijkstra’s algorithm finds shortest paths from a start vertex $s$ to all the other vertices

- It works on a simple graph with non-negative weights (i.e., it works only if $w(e) \geq 0$, for all edges $e$)

Dijkstra’s algorithm

- The algorithm computes for each vertex $u$ the \textit{distance} to $u$ from the start vertex $s$, that is, the weight of a shortest path between $s$ and $u$

- The algorithm keeps track of the set of vertices for which the distance has been computed, called the \textit{cloud} $S$
Dijkstra’s algorithm

• Every vertex has a *label* associated with it
• For any vertex *u*, we can refer to its *d* label as *d*[[*u*]]
• *d*[[*u*]] stores an approximation of the distance between *s* and *u*
• The algorithm will update a *d*[ [*u* ]] value when it finds a shorter path from *s* to *u*

Dijkstra’s algorithm

• When a vertex *u* is added to the cloud, its label *d*[ [*u* ]] is equal to the actual (final) distance between the starting vertex *s* and vertex *u*
• Initially, we set
  – *d*[ [*s* ]] = 0 ... the distance from *s* to itself is 0 ...
  – *d*[ [*u* ]] = ∞ for *u* ≠ *s* ... these will change ...
Edge relaxation

- For each vertex $v$ in the graph, we maintain in $d[v]$ the estimate of the shortest path from $s$
- **Relaxing** an edge $(u,v)$ means testing whether we can improve the shortest path to $v$ found so far by going through $u$

```
Relax(u,v)
```

```
5 2 9
5 2 7
```

```
5 2 6
```

Expanding the Cloud

- Repeat until all vertices have been put in the cloud
  - let $u$ be a vertex not in the cloud that has smallest $d[u]$
    (on the first iteration, the starting vertex will be chosen)
  - we add $u$ to the cloud $S$
  - we update $d[.]$ of the adjacent vertices of $u$ as follows
    *(edge relaxation)*
    
    **for** each vertex $z$ adjacent to $u$ **do**
    
    **if** $z$ is not in the cloud $S$ **then**
    
    
    **if** $d[u] + \text{weight}(u,z) < d[z]$ **then**
    
    
    $d[z] \leftarrow d[u] + \text{weight}(u,z)$
Example $s=\text{BWI}$

(c)
Example

Example

(e)

(f)

(g)

(h)
Example

Dijkstra’s algorithm

Algorithm ShortestPath(G,ν):

Input: A simple undirected weighted graph G with nonnegative edge weights,
and a distinguished vertex ν of G

Output: A label D[ν], for each vertex u of G, such that D[u] is the distance from ν to u in G

Initialize D[ν] ← 0 and D[u] ← +∞ for each vertex u ≠ ν.
Let a priority queue Q contain all the vertices of G using the D labels as keys.

while Q is not empty do

{pull a new vertex u into the cloud}

u ← Q.removeMin()

for each vertex z adjacent to u such that z is in Q do

{perform the relaxation procedure on edge (u,z)}

if D[u] + w((u,z)) < D[z] then

D[z] ← D[u] + w((u,z))

Change to D[z] the key of vertex z in Q.

return the label D[u] of each vertex u

D[.] is d[.]
Time complexity

- Use a heap-based priority queue $Q$ to store the vertices not in the cloud, where $d[u]$ is the key of a vertex $u$ in $Q$
- Insert all vertices in $Q$, takes $O(n \log n)$
- Each iteration of the while, we spend $O(\log n)$ time to remove vertex $u$ from $Q$ and $O(\text{deg}(u) \log n)$ to perform the relaxation step
- Overall, $O(n \log n + \sum_v (\text{deg}(v) \log n))$ which is $O((n+m) \log n)$ [using binary heaps]

Greedy choice

- **Theorem:** In Dijkstra’s algorithm, whenever a vertex $u$ is pulled into $S$, the label $d[u]$ is equal to $d(s,u)$ (the length of a shortest path from $s$ to $u$), and the equality is maintained thereafter
- **Proof:** (by contradiction) omitted
Negative weights

- Dijkstra fails on graphs with negative edges
- **Example:** Bringing $z$ into $C$ and performing edge relaxation invalidates the previously computed shortest path distance (124) to $x$

![Diagram of a graph with nodes and edges labeled with weights.]

Bellman-Ford’s algorithm
Bellman-Ford’s algorithm

• Dijkstra’s algorithm does not work when the weighted graph contains negative edges
  – we cannot be greedy anymore on the assumption that the lengths of paths will not decrease in the future
• Bellman-Ford’s algorithm detects negative cycles (returns false) or returns the shortest path-tree

Bellman-Ford’s algorithm

• Use $d[?]$ labels (like in Dijkstra’s and Prim’s)
• Initialize $d[s]=0$, $d[?]=\infty$ otherwise
• Perform $|V|-1$ rounds
• In each round, we attempt an edge relation for all the edges in the graph (arbitrary order)
• An extra round of edge relaxation can tell the presence of a negative cycle
Bellman-Ford’s algorithm

Algorithm Bellman-Ford \((G(V,E), s)\)

\[
\text{for each } u \text{ in } V \\
d[u] \leftarrow \infty \\
d[s] \leftarrow 0 \\
\text{for } i \leftarrow 1 \text{ to } |V|-1 \text{ do} \\
\quad \text{for each } (u,v) \text{ in } E \text{ do} \\
\quad \quad \text{if } d[v] > d[u] + w(u,v) \text{ then} \\
\quad \quad \quad d[v] \leftarrow d[u] + w(u,v) \\
\quad \text{for each } (u,v) \text{ in } E \text{ do} \\
\quad \quad \text{if } d[v] > d[u] + w(u,v) \text{ then} \\
\quad \quad \quad \text{return } FALSE \\
\text{return } d[], TRUE
\]

Iteration 0
Iteration 3

Iteration 4
Observe that BF is essentially dynamic programming. Let \( d(i, j) \) = “cost of the shortest path from \( s \) to \( i \) that uses at most \( j \) edges/hops”

\[
d(i, j) = \begin{cases} 
0 & \text{if } i = s \& j = 0 \\
\infty & \text{if } i \neq s \& j = 0 \\
\min_{(k,j) \in E} \{d(k, j-1) + w(k, i), d(i, j-1)\} & \text{if } j > 0
\end{cases}
\]

Why \( O(nm) \)?

### Bellman-Ford’s correctness

**Theorem 7.4:** If after performing the above computation there is an edge \((u, z)\) that can be relaxed (that is, \(D[u] + w((u, z)) < D[z]\)), then the graph \(G\) contains a negative-weight cycle. Otherwise, \(D[u] = d(v, u)\) for each vertex \(u\) in \(G\).

- Works for negative-weight edges
- Can detect the presence of negative-weight cycles
- Running time is \(O(nm)\)
Floyd-Warshall’s algorithm

All-pairs shortest path

• We want to compute the shortest path distance between every pair of vertices in a directed graph $G$ ($n$ vertices, $m$ edges)

• We want to know $D[i,j]$ for all $i,j$, where $D[i,j]=$ shortest distance from $v_i$ to $v_j$
All-pairs shortest path

• If $G$ has no negative-weight edges, we could use Dijkstra’s algorithm repeatedly from each vertex

• It would take $O(n (m+n) \log n)$ time, that is $O(n^2 \log n + nm \log n)$ time, which could be as large as $O(n^3 \log n)$

All-pairs shortest path

• If $G$ has negative-weight edges (but no negative-weight cycles) we could use Bellman-Ford’s algorithm repeatedly from each vertex

• Recall that Bellman-Ford’s algorithm runs in $O(nm)$

• It would take $O(n^2 m)$ time, which could be as large $O(n^4)$ time
All-pairs shortest path

• We now see an algorithm to solve the all-pairs shortest path in $O(n^3)$ time

• The graph can contain negative-weight edges (but no negative-weight cycles)

All-pairs shortest path

• Let $G=(V,E)$ a weighted directed graph

• Let $V=(v_1,v_2,\ldots,v_n)$

• Define cost function $D_{i,j}^k = "the shortest distance from $v_i$ to $v_j$ using only vertices $\{v_1, v_2, \ldots, v_k\}"$
A dynamic programming shortest-path

Initially we set

\[ D_{i,j}^0 = \begin{cases} 
0 & \text{if } i = j \\
\infty & \text{otherwise} \\
w((v_i, v_j)) & \text{if } (v_i, v_j) \in E
\end{cases} \]
A dynamic programming shortest-path

\[ D^k_{i,j} = \min \{ D^{k-1}_{i,j}, D^{k-1}_{i,k} + D^{k-1}_{k,j} \}. \]
All-pairs shortest path

Algorithm $\text{AllPairs}(\bar{G})$:

**Input:** A weighted directed graph $\bar{G}$ with $n$ vertices numbered $v_1, v_2, \ldots, v_n$

**Output:** A matrix $D$ such that $D[i, j]$ is distance from $v_i$ to $v_j$ in $\bar{G}$

for $i \leftarrow 1$ to $n$ do
  for $j \leftarrow 1$ to $n$ do
    if $i = j$ then
      Set $D^0[i, i] \leftarrow 0$ and continue looping
    if $(v_i, v_j)$ is an edge in $\bar{G}$ then
      Set $D^0[i, j] \leftarrow w((v_i, v_j))$
    else
      Set $D^0[i, j] \leftarrow +\infty$
  
for $i \leftarrow 1$ to $n$ do
  for $j \leftarrow 1$ to $n$ do
    for $k \leftarrow 1$ to $n$ do
      Set $D^k[i, j] \leftarrow \min\{D^{k-1}[i, j], D^{k-1}[i, k] + D^{k-1}[k, j]\}$

Return $D^n$

Floyd-Warshall’s algorithm computes the shortest path distance between each pair of vertices of $G$ in $O(n^3)$ time
Minimum Spanning Tree

• Given a weighted undirected graph $G$, find a tree $T$ that spans all the vertices of $G$ and minimizes the sum of the weights on the edges, that is

$$w(T) = \sum_{e \in T} w(e)$$

• We want a spanning tree of minimum cost
Example

\[ w(T) = 4 + 8 + 7 + 9 + 2 + 4 + 2 + 1 = 37 \]

Note that the MST is not necessarily unique

For example, add \((a, h)\), delete \((b, c)\)

Growing a MST: Generic algorithm

- Grow MST one edge at a time
- Manage a set of edges \(A\), maintaining the following invariant
  - prior to each iteration, \(A\) is a subset of some MST
- At each iteration, we determine an edge \((u, v)\) that can be added to \(A\) without violating this invariant
- If \(A \cup \{(u, v)\}\) is also a subset of a MST, then \((u, v)\) is called a safe edge for \(A\)
Generic MST algorithm

\textsc{generic-mst}(G, w)
1 \ A \leftarrow \emptyset \\
2 \ \textbf{while} \ A \text{ does not form a spanning tree} \\
3 \ \quad \textbf{do} \ \text{find an edge } (u, v) \text{ that is safe for } A \\
4 \quad A \leftarrow A \cup \{(u, v)\} \\
5 \ \textbf{return} \ A

- Loop in lines 2-4 is executed \(|V| - 1\) times because any MST tree contains \(|V| - 1\) edges
- The overall execution time depends on how to find a safe edge (step 3)

First Edge

- Which edge is clearly safe? Is the “shortest edge” safe?
Greedy Choice

- Definitions
  - Cut \((S, V-S)\): a partition of \(V\)
  - Crossing edge: one endpoint in \(S\) and the other in \(V-S\)
  - A cut respects a set of \(A\) of edges if no edges in \(A\) crosses the cut
  - A light edge crossing a partition if its weight is the minimum of any edge crossing the cut

- Theorem. Let \(A\) be a subset of \(E\) that is included in some MST of \(G=(V,E)\). Let \((S, V-S)\) be any cut of \(G\) that respects \(A\), and let \((u, v)\) be a light edge crossing \((S, V-S)\). Then, edge \((u, v)\) is safe for \(A\).

Examples of Cuts and light edges

Figure 23.2 Two ways of viewing a cut \((S, V-S)\) of the graph from Figure 23.1. (a) The vertices in the set \(S\) are shown in black, and those in \(V-S\) are shown in white. The edges crossing the cut are those connecting white vertices with black vertices. The edge \((d, c)\) is the unique light edge crossing the cut. A subset \(A\) of the edges is shaded; note that the cut \((S, V-S)\) respects \(A\), since no edge of \(A\) crosses the cut. (b) The same graph with the vertices in the set \(S\) on the left and the vertices in the set \(V-S\) on the right. An edge crosses the cut if it connects a vertex on the left with a vertex on the right.
Proof of Greedy Choice Thm

- Let $T$ be a MST that includes $A$, and assume $T$ does not contain the light edge $(u, v)$. [If it does, we are done.]
- First, we construct another MST $T'$ that includes $A \cup \{(u, v)\}$
  - Adding $(u, v)$ to $T$ induces a cycle
    - Let $(x, y)$ be the edge on the cycle crossing $(S, V-S)$, then $w(u,v) \leq w(x,y)$
    - $T' = T - (x, y) \cup (u, v)$
    - $T'$ is also a MST because it is a spanning tree of $G$ and $w(T') = w(T) - w(x,y) + w(u,v) \leq w(T)$
- Second, we prove that $(u, v)$ is safe for $A$
  - Since $A \subseteq T$ and $(x, y) \notin A$ then $A \subseteq T'$. Therefore $A \cup \{(u, v)\} \subseteq T'$. Since $T'$ is a MST, $(u,v)$ is safe for $A$

Optimal substructure property

- Let $T$ be an MST of $G$. Let $(u,v)$ be an edge in $T$
- Removing $(u,v)$ partitions $T$ into two trees $T_1$ and $T_2$
- Let $(S, V-S)$ be a cut that respect $T_1$, let $E_1$ be the subset of edges incident to $S$, and $E_2$ be the subset of edges incident to $V-S$
- Claim: $T_1$ is an MST of $G_1 = (S, E_1)$, and $T_2$ is an MST of $G_2 = (V-S, E_2)$
  - Note that $w(T) = w(u,v) + w(T_1) + w(T_2)$
  - A “cheaper” tree than $T_1$ or $T_2$ cannot exists, otherwise $T$ would not be optimal
Generic MST algorithm

\text{GENERIC-MST}(G, w)
1. \text{\textbf{A} } \leftarrow \emptyset
2. \textbf{while } \text{A does not form a spanning tree}
3. \quad \textbf{do } \text{find an edge } (u, v) \text{ that is safe for } A
4. \quad \text{\textbf{A} } \leftarrow \text{\textbf{\textbf{A}}} \cup \{(u, v)\}
5. \textbf{return } \textbf{A}

Kruskal’s algorithm
Kruskal’s algorithm

- Consider the edges one at a time, by increasing weight

- Accept an edge if it connects two different trees

Example
Example

(c)

(d)

Example

(e)

(f)
Example

Example

(i)

(j)
Example

(k)

(l)

Example

(m)

(n)
Kruskal’s algorithm

Algorithm Kruskal\((G)\):

Input: A simple connected weighted graph \(G\) with \(n\) vertices and \(m\) edges
Output: A minimum spanning tree \(T\) for \(G\)

for each vertex \(v\) in \(G\) do

Define an elementary cluster \(C(v) = \{v\}\).
Initialize a priority queue \(Q\) to contain all edges in \(G\), using the weights as keys.
\(T \leftarrow \emptyset\) \{ \(T\) will ultimately contain the edges of the MST\}

while \(T\) has fewer than \(n - 1\) edges do

\((u,v) \leftarrow Q\text{-removeMin()}\)

Let \(C(v)\) be the cluster containing \(v\), and let \(C(u)\) be the cluster containing \(u\).
if \(C(v) \neq C(u)\) then

Add edge \((v,u)\) to \(T\).
Merge \(C(v)\) and \(C(u)\) into one cluster, that is, union \(C(v)\) and \(C(u)\).

return tree \(T\)

Data Structure for Kruskal’s algorithm

- The data structure maintains a forest of trees
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets, with the following operations
  - \(\text{find}(u)\): return the set storing \(u\)
  - \(\text{union}(u,v)\): replace the sets storing \(u\) and \(v\) with their union
Data structure for sets

A={1,4,7}     B={2,3,6,9}    C={5,8,10,11,12}

Representation of a Partition

- Each set is stored in a sequence (list)
- Each element has a reference back to the set
  - operation \textit{find}(u) takes $O(1)$ time, and returns the set of which $u$ is a member
  - in operation \textit{union}(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation \textit{union}(u,v) is $\min(n_u,n_v)$, where $n_u$ and $n_v$ are the sizes of the sets storing $u$ and $v$
Kruskal’s algorithm running time

• Whenever a vertex is added to a tree, the size of the tree containing the vertex at least double
• Each vertex is moved to a new tree at most $\log n$ times
• Total time merging trees is $O(n \log n)$
• Cost of creating the priority queue $O(m \log m)$ which is $O(m \log n)$
• Overall running time is $O((n+m) \log n)$

Prim’s algorithm
Prim’s algorithm

- The edges in the set $A$ always forms a single tree
- The tree starts from an arbitrary vertex and grows until the tree spans all the vertices in $V$
- At each step, a light edge is added to the tree $A$ that connects $A$ to an isolated vertex of $G_A=(V, A)$
- “Greedy” because the tree is augmented at each step with an edge that contributes the minimum amount possible to the tree’s weight

Prim’s vs. Dijkstra’s

- Prim’s strategy similar to Dijkstra’s
- Grows the MST $T$ one edge at a time
- Cloud covering the portion of $T$ already computed
- Label $D[u]$ associated with each vertex $u$ outside the cloud (distance to the cloud)
Prim’s algorithm

• For any vertex $u$, $D[u]$ represents the weight of the current best edge for joining $u$ to the rest of the tree in the cloud (as opposed to the total sum of edge weights on a path from start vertex to $u$)
• Use a priority queue $Q$ whose keys are $D$ labels, and whose elements are vertex-edge pairs

Prim’s algorithm

• Any vertex $v$ can be the starting vertex
• We still initialize $D[v]=0$ and all the $D[u]$ values to $+\infty$

• We can reuse code from Dijkstra’s, just change a few things
Example

(a)

Example

(c)

(d)
Example

(g)

(h)

125
Example

Pseudo Code

Algorithm PrimJarnik(G):

Input: A weighted connected graph G with n vertices and m edges
Output: A minimum spanning tree T for G

Pick any vertex v of G
D[v] = 0
for each vertex u ≠ v do
D[u] ← +∞
Initialize T ← Ø.
Initialize a priority queue Q with an item ((u,null),D[u]) for each vertex u, where (u,null) is the element and D[u] is the key.
while Q is not empty do
(u,e) ← Q.removeMin()
Add vertex u and edge e to T.
for each vertex z adjacent to u such that z is in Q do
perform the relaxation procedure on edge (u,z)
if w((u,z)) < D[z] then
D[z] ← w((u,z))
Change to (z,(u,z)) the element of vertex z in Q.
Change to D[z] the key of vertex z in Q.

return the tree T
Time complexity

- Initializing the queue takes $O(n \log n)$ [binary heap]
- Each iteration of the while, we spend $O(\log n)$ time to remove vertex $u$ from $Q$ and $O(\text{deg}(u) \log n)$ to perform the relaxation step
- Overall, $O(n \log n + \sum_v (\text{deg}(v) \log n))$ which is $O((n+m) \log n)$ [if using a binary heap]

Summary

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time complexity</th>
<th>Notes</th>
</tr>
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<tbody>
<tr>
<td>Dijkstra</td>
<td>$O((n+m) \log n)$ using p.q.\n$O(n^2+m)$ using array</td>
<td>Non-negative weights</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>$O(mn)$</td>
<td>Negative weights ok</td>
</tr>
<tr>
<td>All-pairs</td>
<td>$O(n^2)$</td>
<td>Negative weights ok</td>
</tr>
<tr>
<td>Shortest path on DAGs</td>
<td>$O(n+m)$</td>
<td>Only for acyclic graphs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Negative weights ok</td>
</tr>
<tr>
<td>Kruskal</td>
<td>$O((n+m) \log n)$ using p.q.</td>
<td></td>
</tr>
<tr>
<td>Prim-Jarnik</td>
<td>$O((n+m) \log n)$ using p.q.</td>
<td></td>
</tr>
</tbody>
</table>
Reading Assignment

• Dasgupta
  – single-source shortest path (4.4, 4.6 and 4.7)
  – DFS on directed graphs, linearization (3.3)
  – all-pairs shortest path (6.6)
  – minimum spanning tree (5.1.3, 5.1.5)