Supported by IDEI Grant PN-II-ID-PCE-2011-3-0981 ”Structure and computational difficulty in combinatorial optimization: an interdisciplinary approach”.

Partition into heapable subsequences, heap tableaux and a multiset extension of Hammersley’s process

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Introduction

CPM: ”All combinatorial problems have applications to computational biology”
THIS PAPER: ”Some problems have applications to Physics of Complex Systems”

Starting Point: Longest Increasing Sequence

3 2 5 7 1 6 9

- Given n (integer) numbers $a_1, a_2, \ldots, a_n$ find the longest subsequence (not necessarily contiguous) that is increasing
- First-year algorithms: Dynamic programming.
- Another (greedy, also first-year) algorithm: Patience sorting.

Start (greedily) building decreasing piles. When not possible, start new pile.

Patience sorting

16, 25, 18, 2, 4, 35, 3, 7, 32, 9, 20

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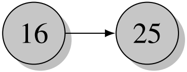

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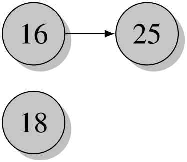

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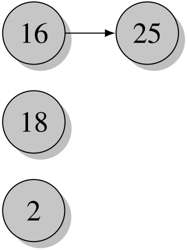

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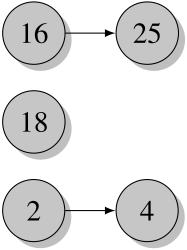

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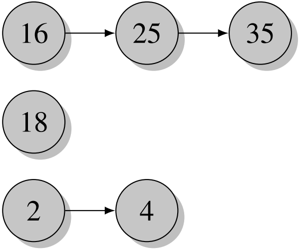

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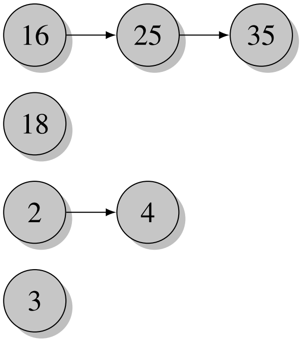

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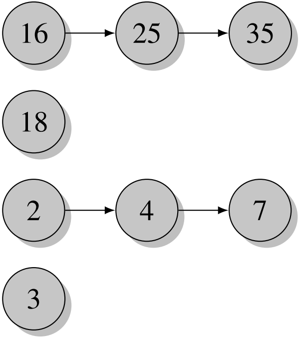

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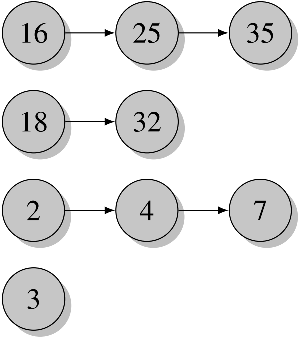

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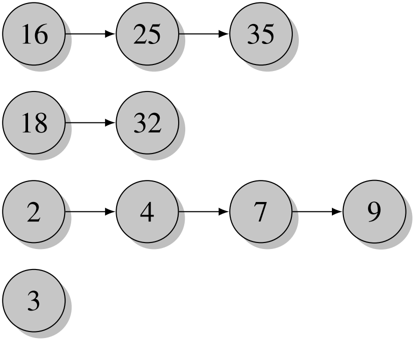

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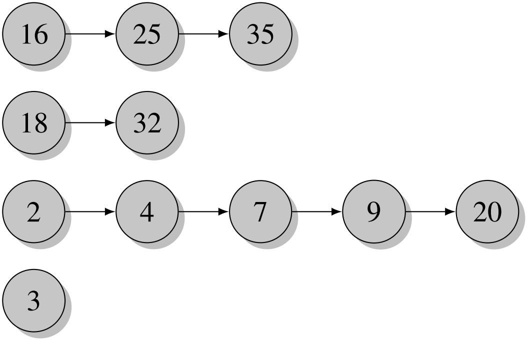

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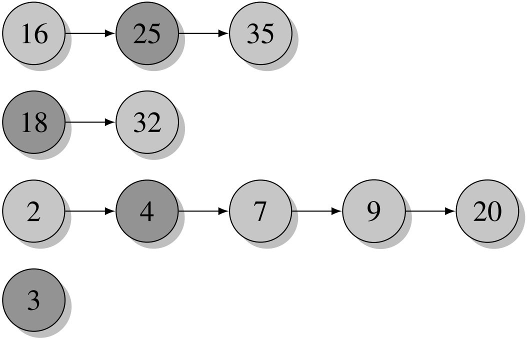
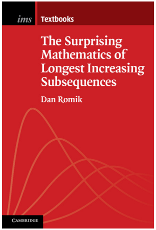


- Partitions the array into increasing (up-)sequences
- SUS(A): the minimal number of such subsequences.
- THEOREM(classical): $SUS(A) = LDS(A)$.

Longest increasing sequence of a random permutation

$$E_{\pi \in S_n}[LIS(\pi)] = 2\sqrt{n} \cdot (1 + o(1)).$$

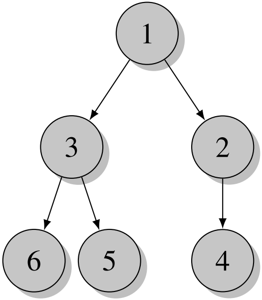
- Logan-Shepp (1977), Veršik-Kerov (1977), Aldous-Diaconis (1995)
- Very rich problem. Connections with nonequilibrium statistical physics and theory of Young tableaux



From data structures to patterns in sequences

Byers, Heeringa, Mitzenmacher, Zervas ANALCO'2011

Sequence of integers A is heapable if it can be inserted into binary heap-ordered tree (not necessarily complete) without having to call HEAPIFY.



Example: 1 3 2 6 5 4   Counterexample: 5 1 . . .

Byers et al. results on heapability

- Polynomial time algorithm for heapability.
- complete heapability: NP-complete
- Longest Heapable Subsequence (LHS): complexity open.
- But with high probability $LHS(\pi) = n - o(n)$, where $\pi \in S_n$ is a random permutation.

Intuition: heapability "weak" versions of increasing sequences.
Recall 1 3 2 6 5 4.
Question (Byers et al.): Does the theory of LIS extend to heapable sequences ?

''Patience heaping''

k-heapable: heapable into a k-ary min-heap
$MHS_k(A)$:Extension of SUS. the smallest number of k-heapable subsequences in a decomposition of A.
Slot of a node: the k (free) positions where children may grow.

Algorithm 1.1: PATIENCE-HEAPING(W)

INPUT $W = (w_1, w_2, \ldots, w_n)$ a list of integers.
Start with empty heap forest $T = \emptyset$.
for i in range(n):
    if (there exists a slot where X_i can be inserted):
        insert X_i in the slot with the lowest value.
  else :
        start a new heap consisting of X_i only.

Patience heaping

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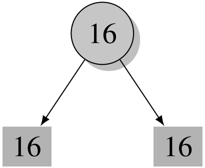

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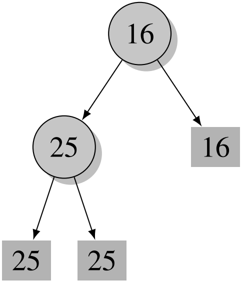

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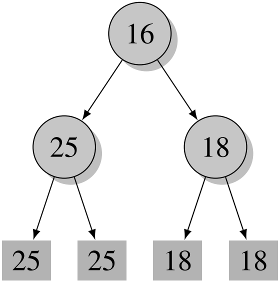

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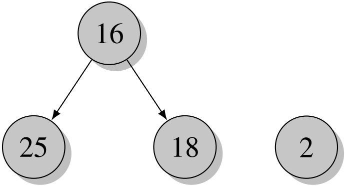

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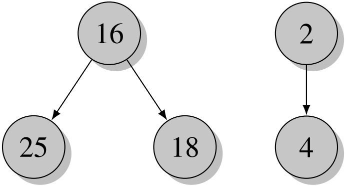

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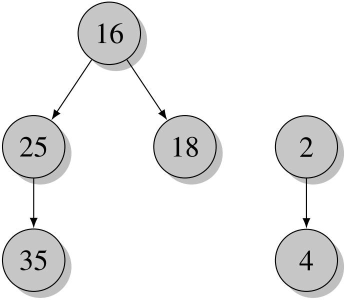

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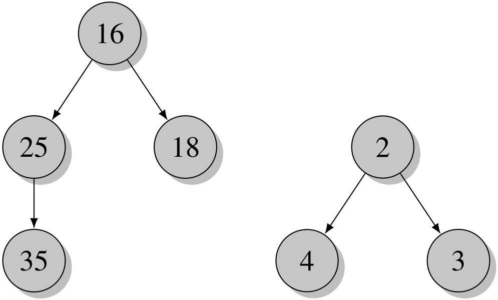

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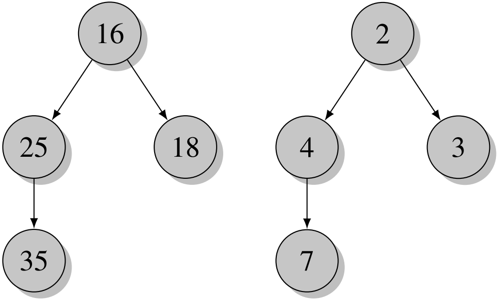

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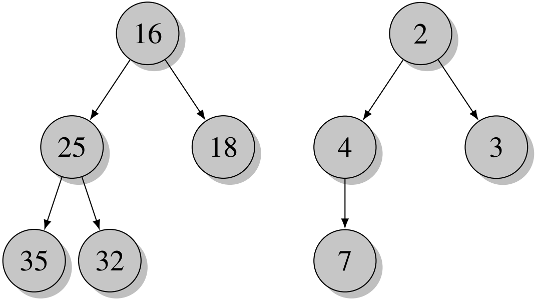

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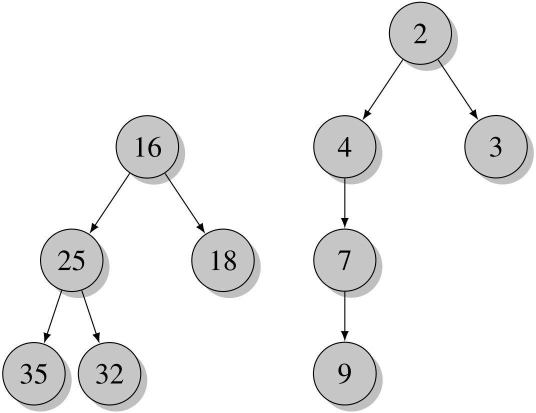

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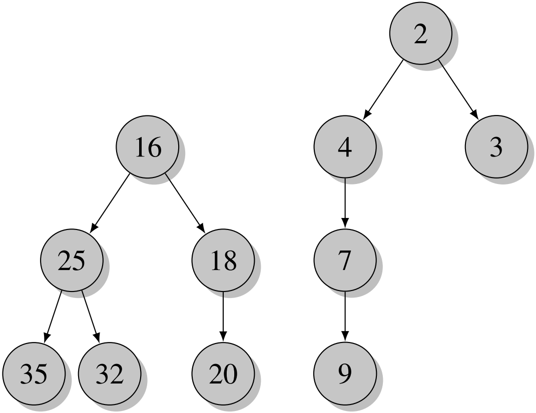


Two (easy) results

THEOREM 1: ”Patience heaping” computes $MHS_k(A)$.
THEOREM 2: (a). there exists sequences X such that
(a). $MHS_k(X) < MHS_{k-1}(X) < \ldots < MHS_1(X)$.
(b). $\sup_X [MHS_{k-1}(X) - MHS_k(X)] = \infty$.

Proof Ideas:

- (a). Define domination relation between multisets of slots.
- Greedy insertion dominates any other insertion+ induction.

If GREEDY creates new heap then any other algorithm does.

- (b). Thm.1 + Example.

Scaling of MHS_k

How does $E[MHS_k(\pi)]$, where π is a random in S_n, behave ?

- Increasing $\equiv$ "1-heapable". $E[LIS(A)] \sim 2\sqrt{n}$.
- For any k growth at least logarithmic.

THEOREM 3: For every fixed $k, n \geq 1$

$$E_{\pi \in S_n}[MHS_k(\pi)] \geq H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \quad (1)$$

the n'th harmonic number, $\sim \ln(n)$.

Proof Idea:

Sequence minima start new heaps.

A beautiful conjecture

CONJECTURE: We have

$$\lim_{n\to\infty} \frac{E[MHS_2[\pi]]}{\ln(n)} = \phi,$$

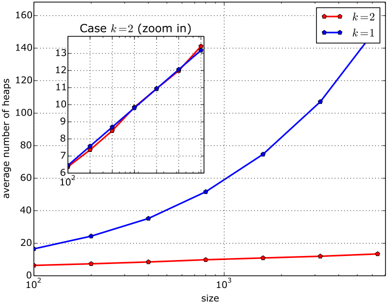
with $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

More generally, for an arbitrary $k \geq 2$ the relevant scaling is

$$\lim_{n\to\infty} \frac{E[MHS_2[\pi]]}{\ln(n)} = \frac{1}{\phi_k}, \tag{2}$$

where ϕ_k is the unique root in $(0, 1)$ of equation
$X^k + X^{k-1} + \ldots + X = 1$.

Evidence: Simulations.



- We kind of know what's going on.
- Can make nonrigorous computations that match experimental predictions
- . . . like physicists do !

We just can't make all steps of the argument rigorous !

Hammersley's process: an interacting particle system.

Tops of piles in patience sorting = live particles in Hammersley's process:

- Particles arive at integer times as random real numbers $X_i \in [0, 1]$.
- Particle X_i kills closest live particle X_j, $X_i < X_j$ (if any)

- studied in the area of interacting particle systems, a field between probability theory and (Nonequilibrium) Statistical Physics.
- relative of a more famous process, the so-called Totally Asymmetric Exclusion Process (TASEP)
- Most illuminating proof of $E[LIS(\pi)] \sim 2\sqrt{n}$ (Aldous-Diaconis) analysis of the so-called hydrodynamic limit of Hammersley's process.

Physics of patience heaping ?

Hammersley's process with k lifelines (HAM_k):

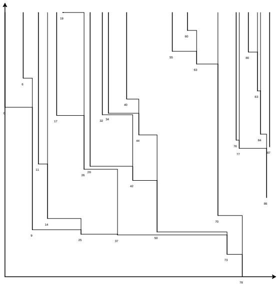
- "Particles" = random numbers $X_i \in [0, 1]$.
- each "particle" is initially endowed with k lives.

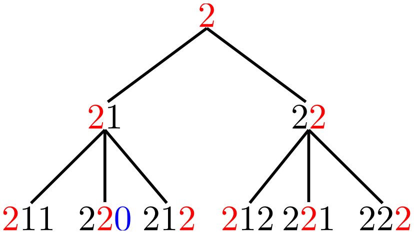
- X_i ~~kills~~ takes one lifeline from closest live X_j, $X_i < X_j$ (if any)

THEOREM 4: At time n the multiset of free slots in patience heaping corresponds (with multiplicities) to live particles in Hammersley's process with k lifelines.

Combinatorial view of process HAM_k

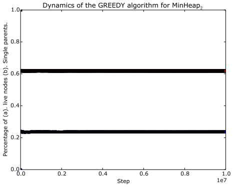
- Words over alphabet $0, 1, 2$ and a (conventional leading) -1.
- Start with $W_0 = -1$ (leftmost marker; not shown below)
- Choose a random position to the right of -1. Put there a 2. Remove 1 from the closest nonzero digit to the right (if any).
- # of heaps = # of insertions that don't remove a lifeline.

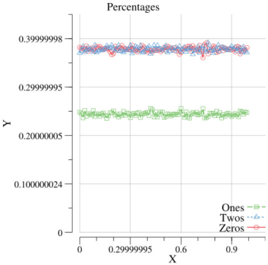




A ”physicist” explanation for the dynamics of HAM_k

```
>>> ===============
RESTART ========
>>>
final[ 1 ]: 0.6179875
0.2359724
final[ 2 ]: 0.6180139
0.2360253
final[ 3 ]: 0.6180207
0.236038
final[ 4 ]: 0.61804
0.2360777
final[ 5 ]: 0.6180516
0.2361006
```





Exp: 5 indep. runs, 10.000.000 steps. Final vals: Fig 1. First trajectory: Fig 2, once every 10.000 steps. Fig 3: binned densities.

- 1. $n \to \infty$: Limit of $W_n =$ compound Poisson process. $W_n =$ random string of 0,1,2 (densities d_0, d_1, d_2).
- 2. Assuming well mixing of digits one can write evolution equations that predict (constant) limit values of d_0, d_1, d_2.
- 3. From this: ”on average” the probability that the number of heaps increases at stage $n \sim \frac{1+\sqrt{5}}{2 \cdot n}$.

Rigorous result on HAM_k

$d_0(n), d_1(n), d_2(n)$: average densities of digits 0,1,2 in word W_n (discarding -1).

FIRST STEP: There exist constants $d_0, d_1, d_2 \in [0, 1]$ such that

$$\lim_{n\to\infty} d_i(n) = d_i, i = 0, 1, 2.$$

Tool: subadditivity/Fekete's Lemma: "If a_n is a sequence with $a_{m+n} \leq a_m + a_n$ then $\lim_{n\to\infty} a_n/n$ exists."

In the paper: Fekete's lemma applies to some suitable independent linear combinations of $d_i(n)$.

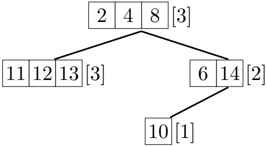
Heapability and heap (Young) tableaux

1	3	5
2	6	
4		

5	3	1
3	1	
1		

- # of fillings: hook formula $\frac{n!}{\prod_i H_i}$ Similar formula for heap-ordered trees (Knuth, TAOCP)
- R-S(K) algorithm: recursive patience sorting, bijection between permutations and pairs of Young tableaux. First row: process Ham_1

Q.: Is there some notion of "heap tableau" that (a). unifies hook formulas for heaps and Young tableaux, and (b). is to heapable seqs/HAM$_k$ process what Young tableaux are for LIS ?



	λ	0	1	00	01	10
1	2	11	6	$\perp$	$\perp$	10
2	4	12	14			
3	8	13				

	λ	0	1	00	01	10
1	6	3	3	$\perp$	$\perp$	1
2	4	2	1			
3	2	1				

- Two ways. we did the easier version [(a), but not (b)]. Partial

Extending the R-S bijection to heap tableaux

I AM OMITTING HERE THE PRESENTATION OF AN ALG. FROM THE PAPER.

- With an additional twist it continues to hold.

THEOREM 7: For every $k \geq 2$ there exists a bijection between permutations $\pi \in S_n$ and pairs (P, Q) of k-heap tableaux with n elements and identical shape, where Q is a standard tableau.

"Q is standard" specific to $k \geq 2$: heaps have "too many degrees of freedom" between siblings. An alg. chooses one specific way.

Case of LIS (Odlyzko & Rains): useful for exactly computing the distribution of $LIS(\pi)$. Perhaps here too.

Many open questions. More about them: (future) journal version.

Thanks !

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Hook formula for Young tableaux

1	3	5
2	6	
4		

5	3	1
3	1	
1		

Figure: (a). Young tableau (b). hook lengths.

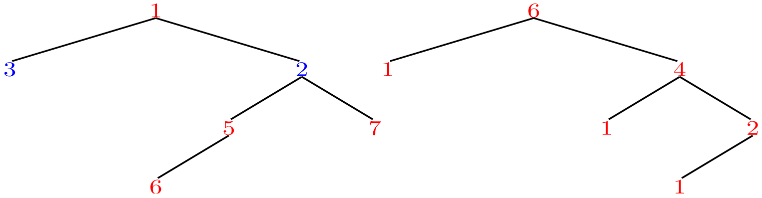
- How many ways to fill up a Young tableau with n cells with numbers from 1 to n ?
- Hook length of cell: number of elements to the right or below it.

Hook formula for Young tableaux (Frame, Robinson, Thrall'54):

$$\frac{n!}{\prod_i H_i}$$

Proof: nontrivial.

Hook formula for heaps



- Hook length of node: number of nodes at or below it.

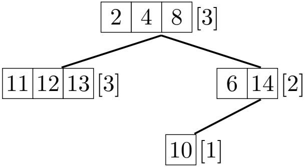
Hook formula for heaps (Knuth, TAOCP exercise in vol. 3):

$$\frac{n!}{\prod_i H_i}$$

Proof: Each node is the smallest in its hook independently with probability $1/H_i$.

Heap tableaux

Is there a common generalization of hook formulas for heaps/Young tableaux ? Is there an object that generalizes both heaps and Young tableaux ?



	λ	0	1	00	01	10
1	2	11	6	$\perp$	$\perp$	10
2	4	12	14			
3	8	13				

	λ	0	1	00	01	10
1	6	3	3	$\perp$	$\perp$	1
2	4	2	1			
3	2	1				

Figure: (a). Heap tableau T_1 and its shape $S(T_1)$ (in brackets) (b). The equivalent Young tableau-like representation of T_1 (c). hook lengths.

Hook inequality for heap tableaux

- Our heap tableaux generalize both heaps and Young tableaux.
- not the version of heap tableaux that corresponds to HAM_k
- Hook formula works for heaps and Young tableaux.
- ... but fails for general heap tableaux.
- perhaps other notion of hook length will do.

THEOREM 6: Given $k \geq 2$ and a k-shape S with n free cells, the number of ways to create a heap tableau T with shape S by filling its cells with numbers $\{1, 2, \ldots, n\}$ is at least

$$\frac{n!}{\prod_{(\alpha,i)\in dom(T)} H_{\alpha,i}}.$$

The bound is tight for Young tableaux, heap-ordered trees, and infinitely many other examples, but is also **not** tight for infinitely many (counter)examples.