Fixed-Parameter Algorithms for Finding Agreement Supertrees

David Fernández-Baca, Sylvain Guillemot, Brad Shutters & Sudheer Vakati
Computer Science Department, Iowa State University

Building Phylogenies from Gene Sequences

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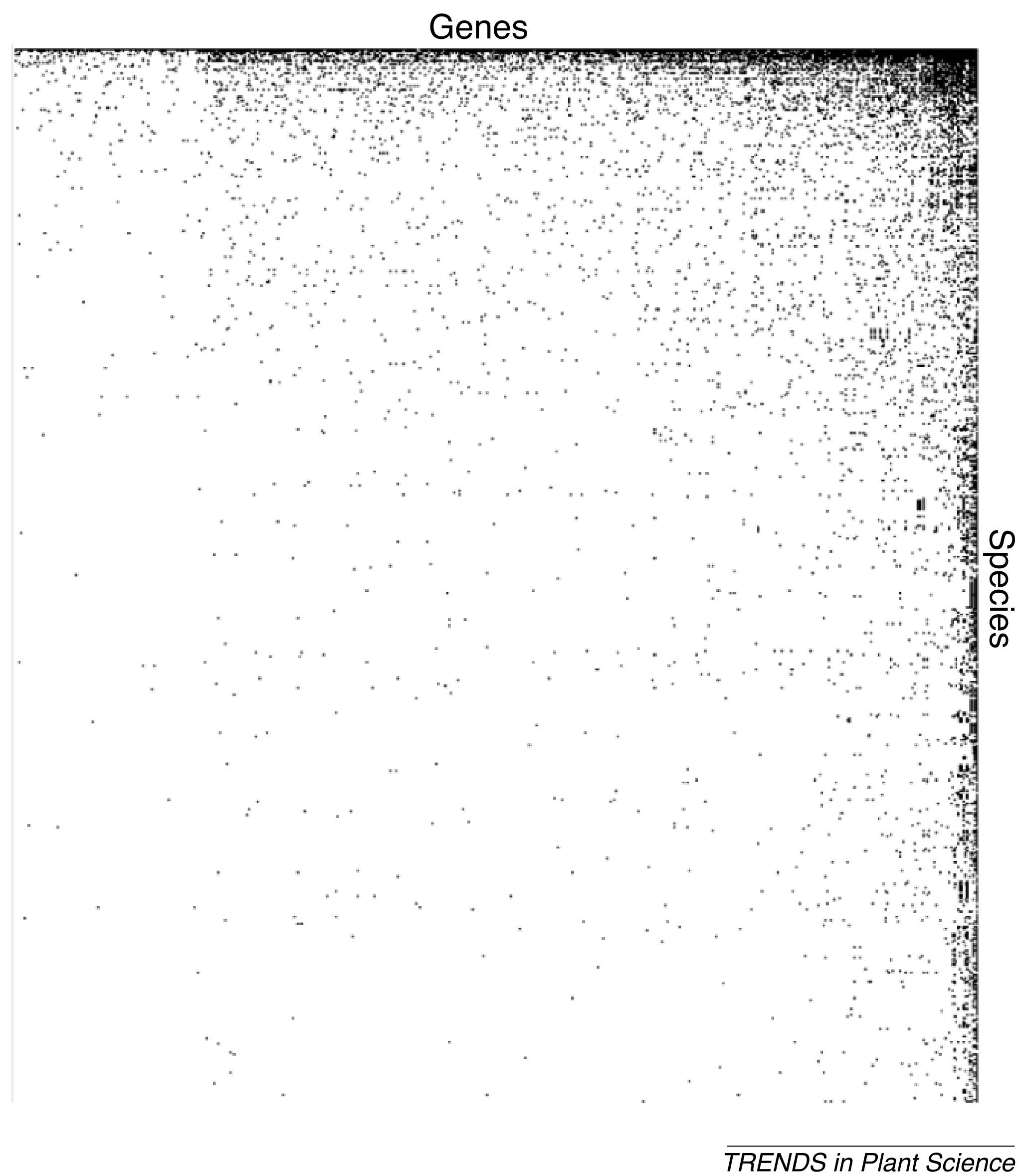
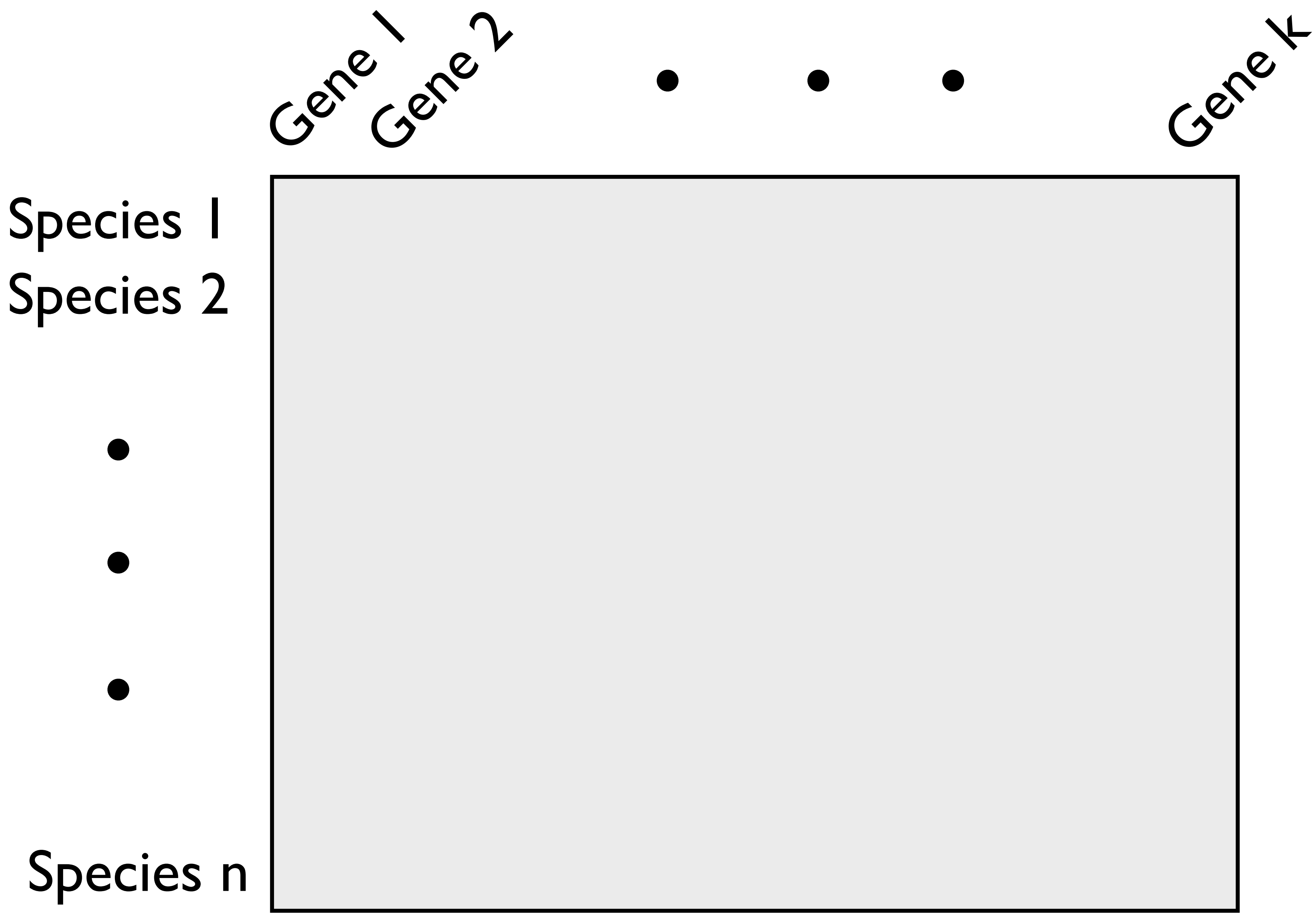
- **Ideal case:** Use genes common to all the taxa (species).
- **Reality:** Low sampling density, number of common genes is very limited.

Low Sampling Density



Sanderson & Driskell, 2003

Supertrees



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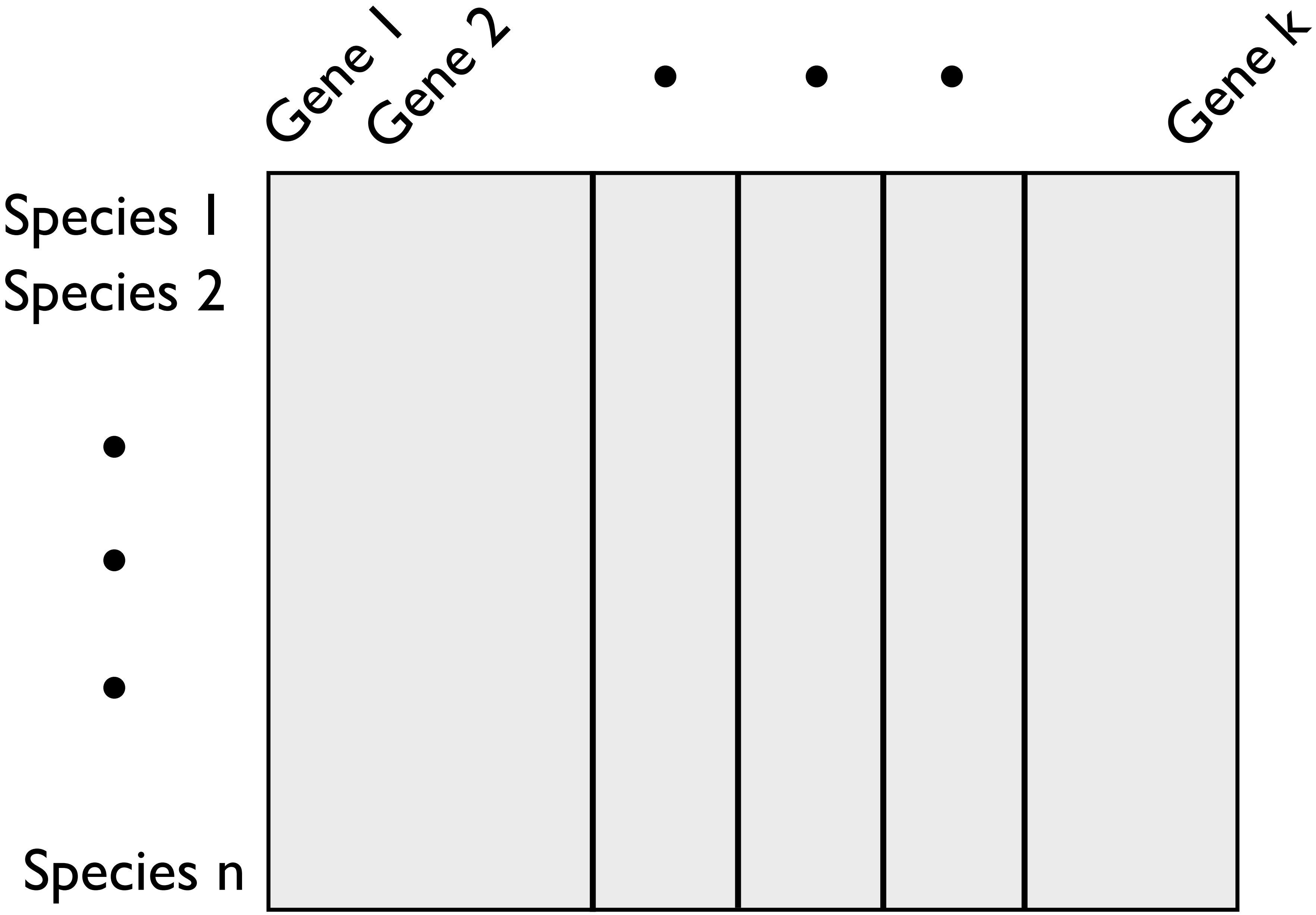

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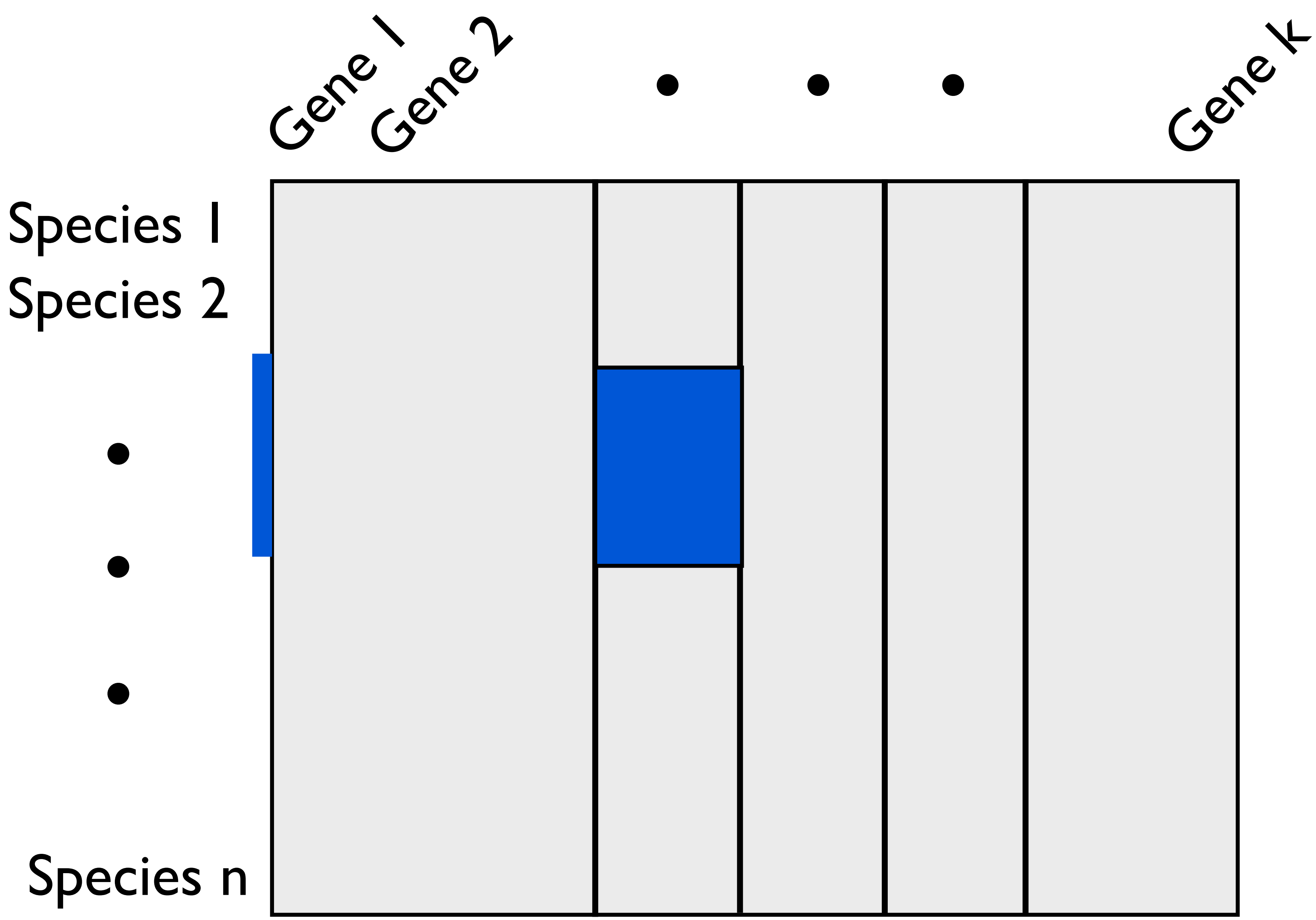

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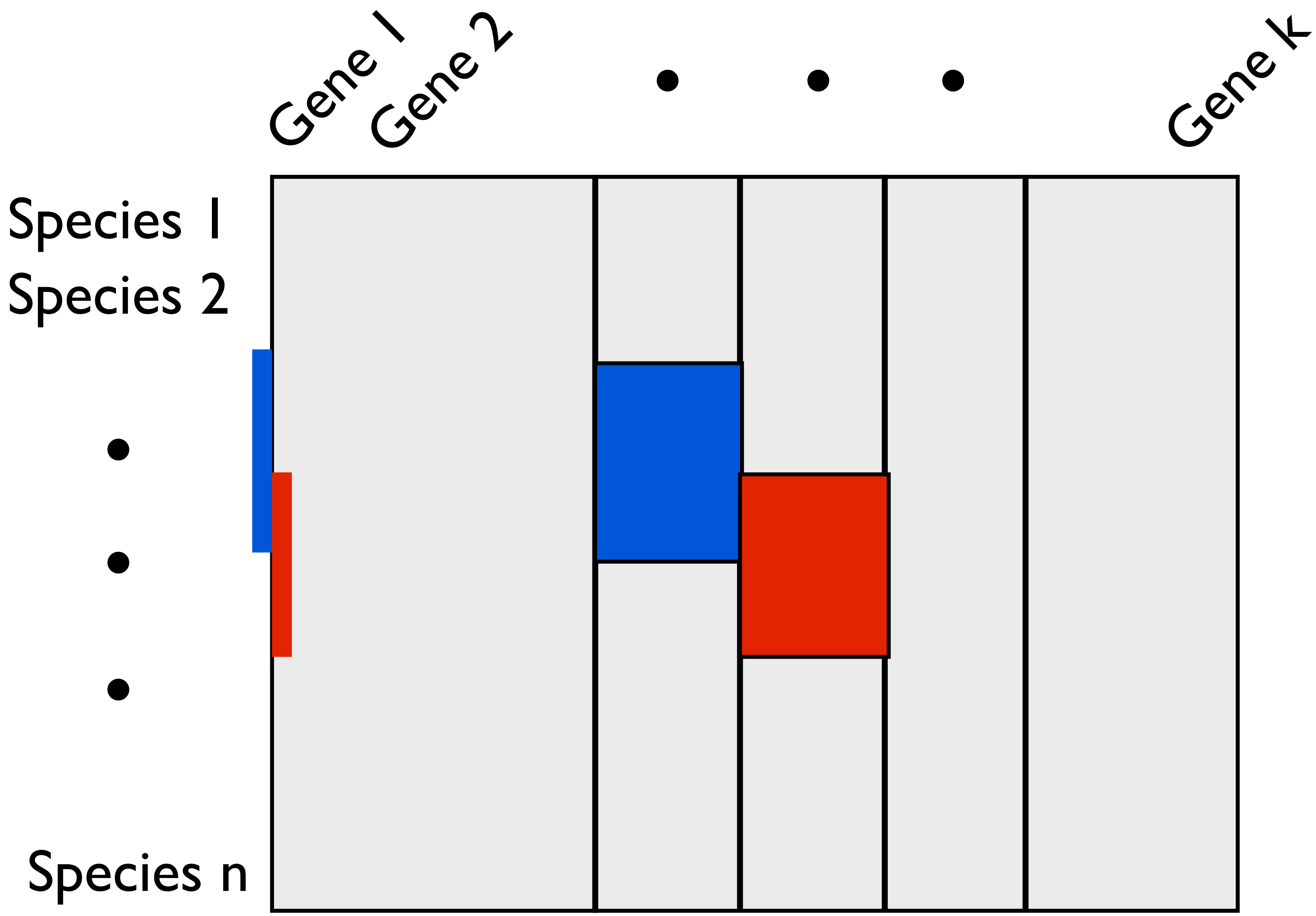

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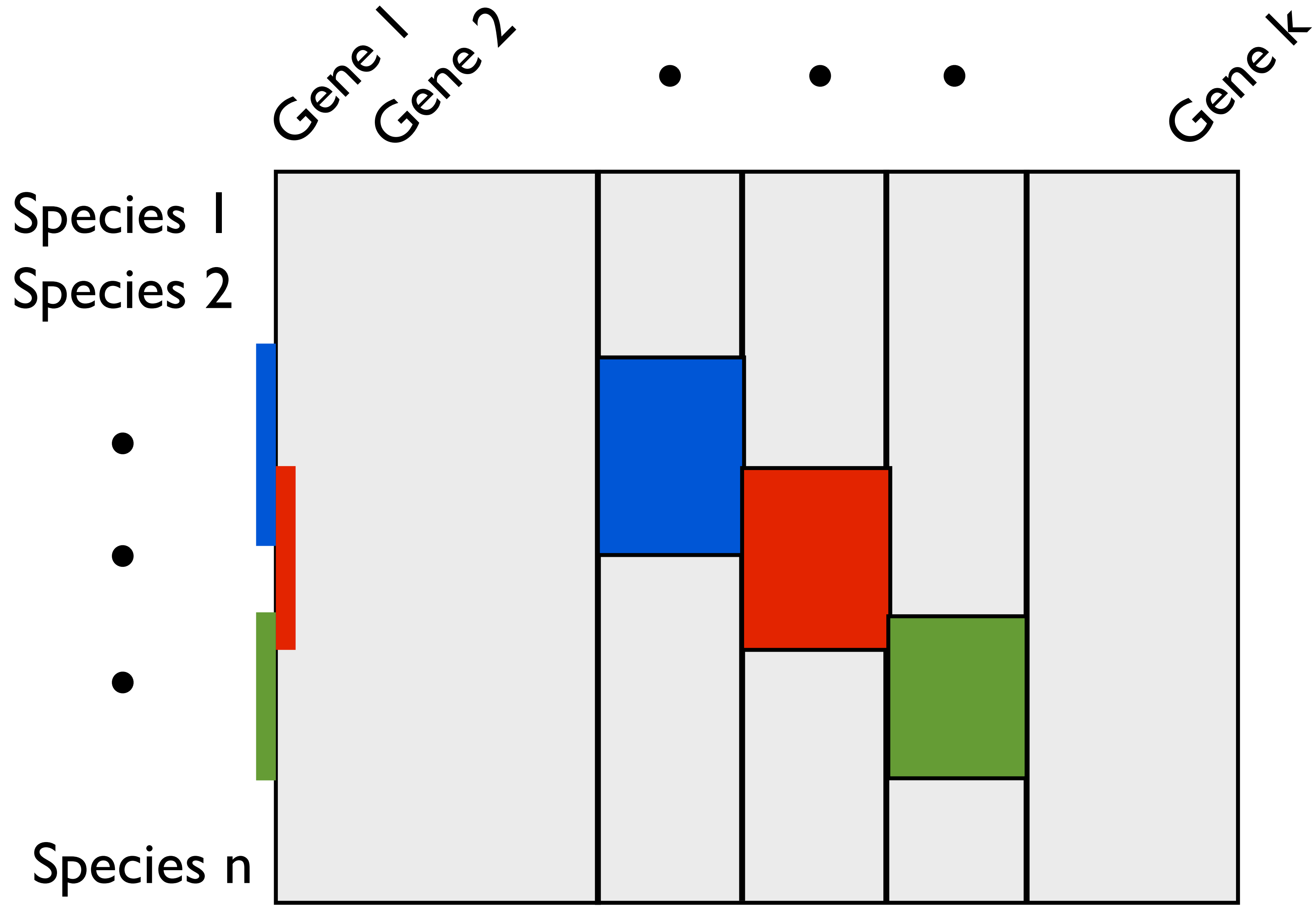

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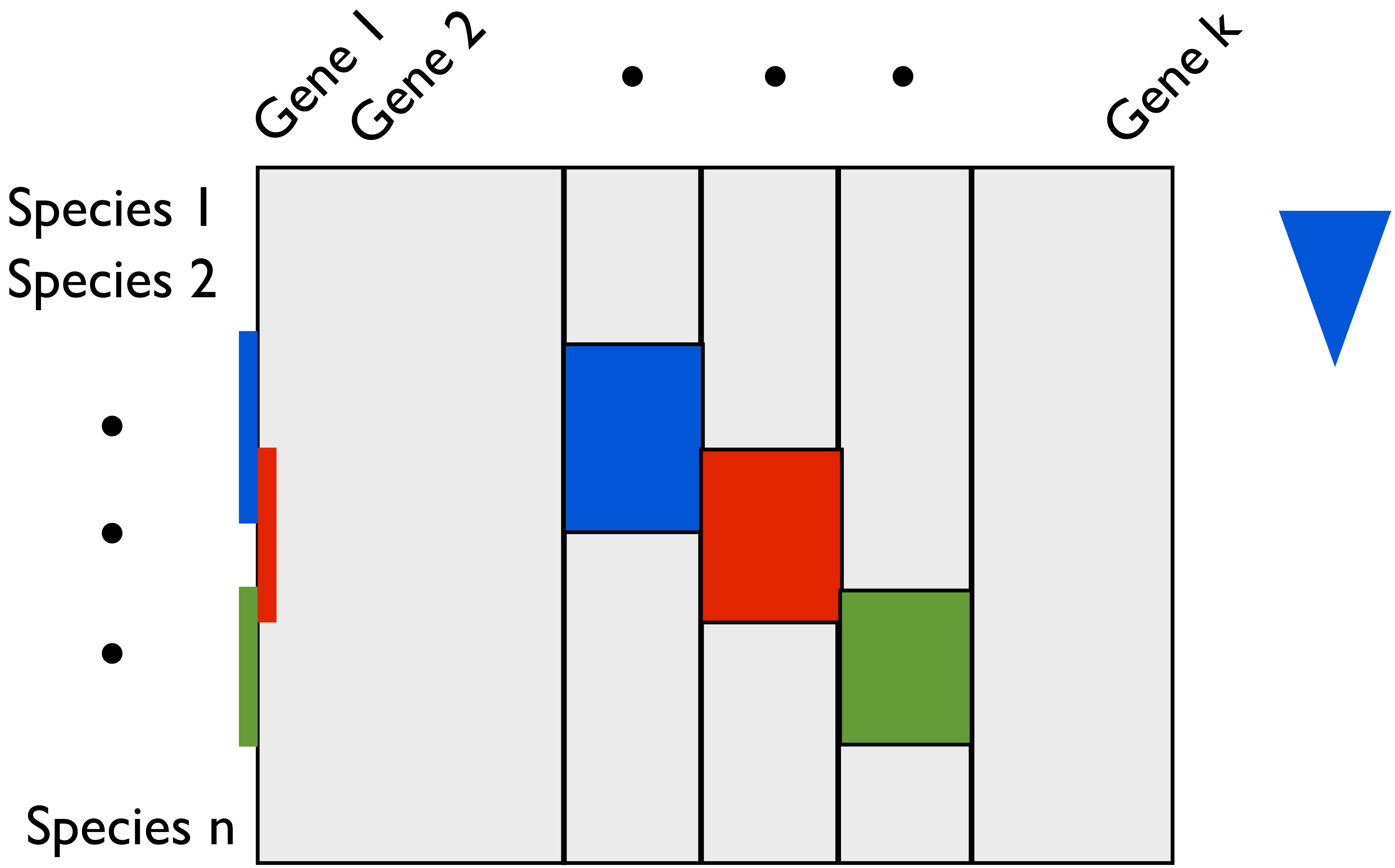

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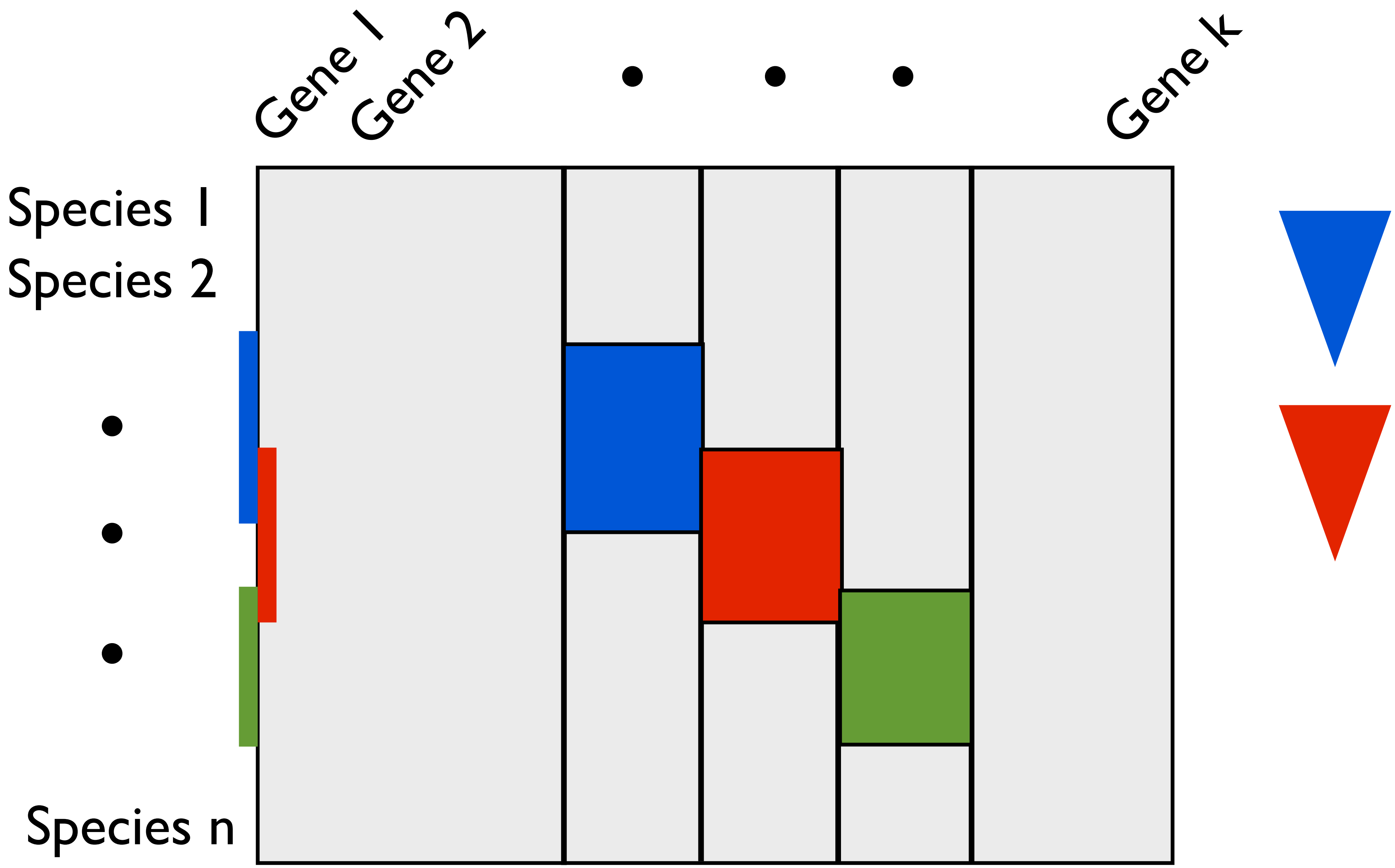

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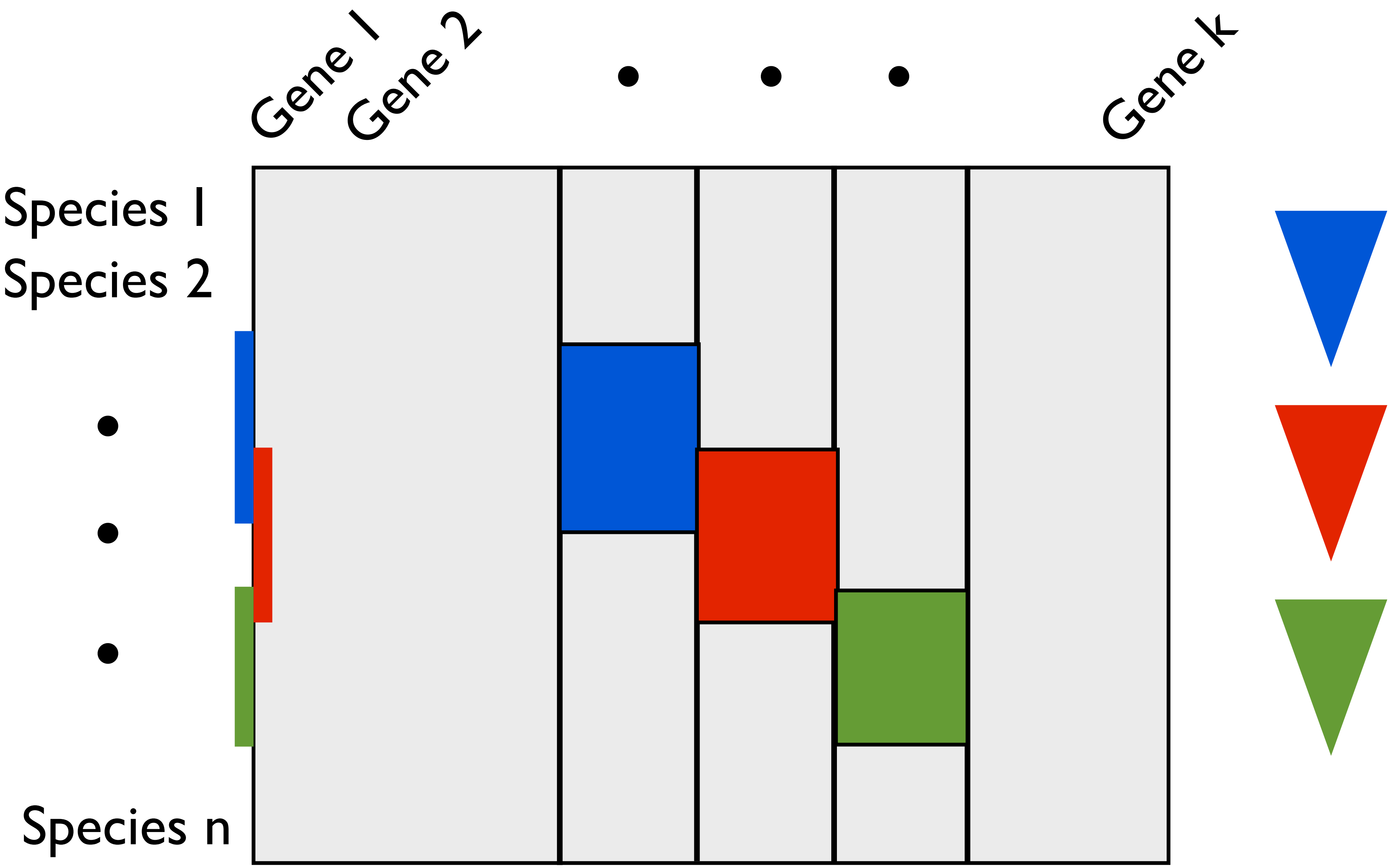

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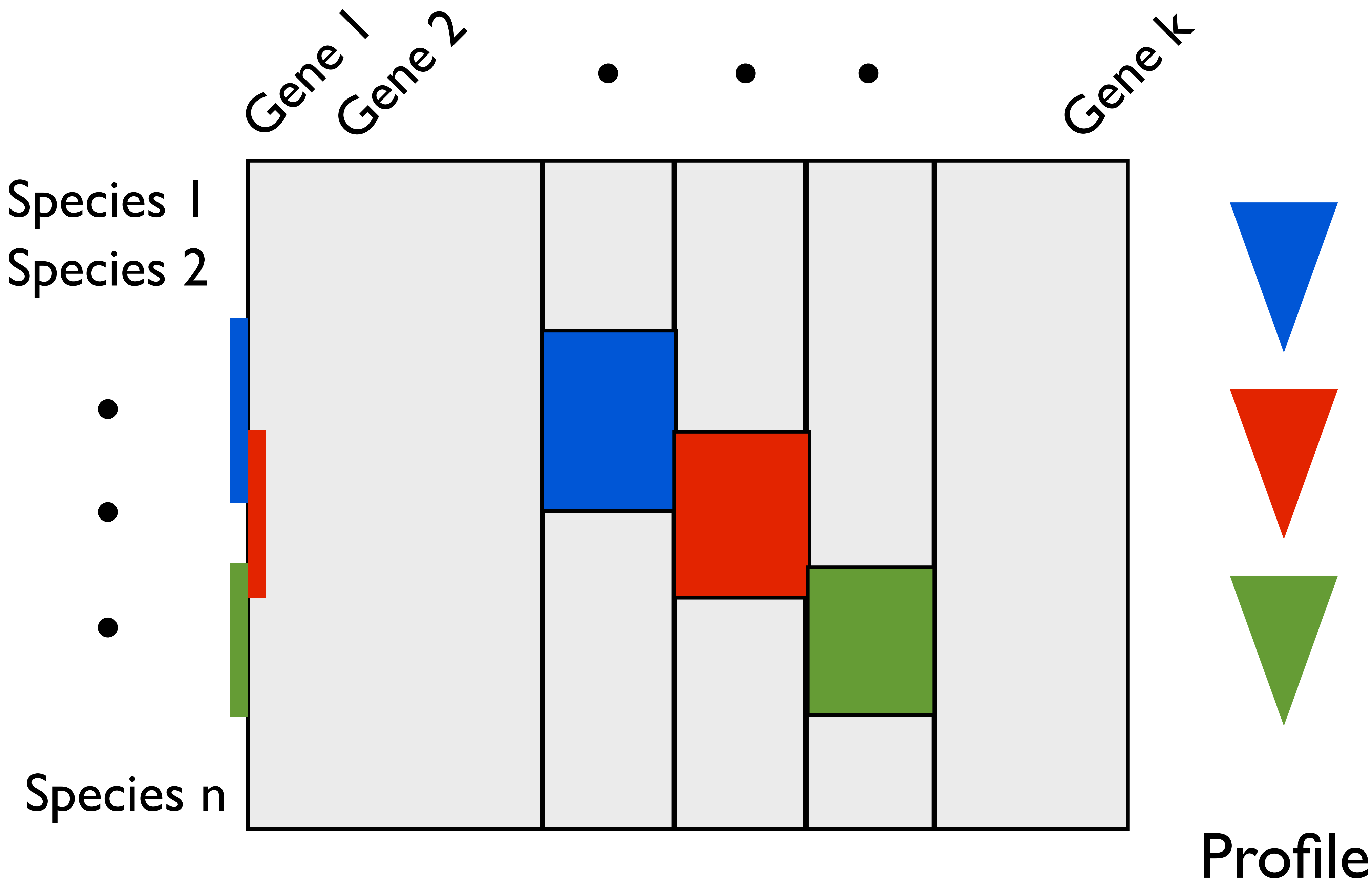

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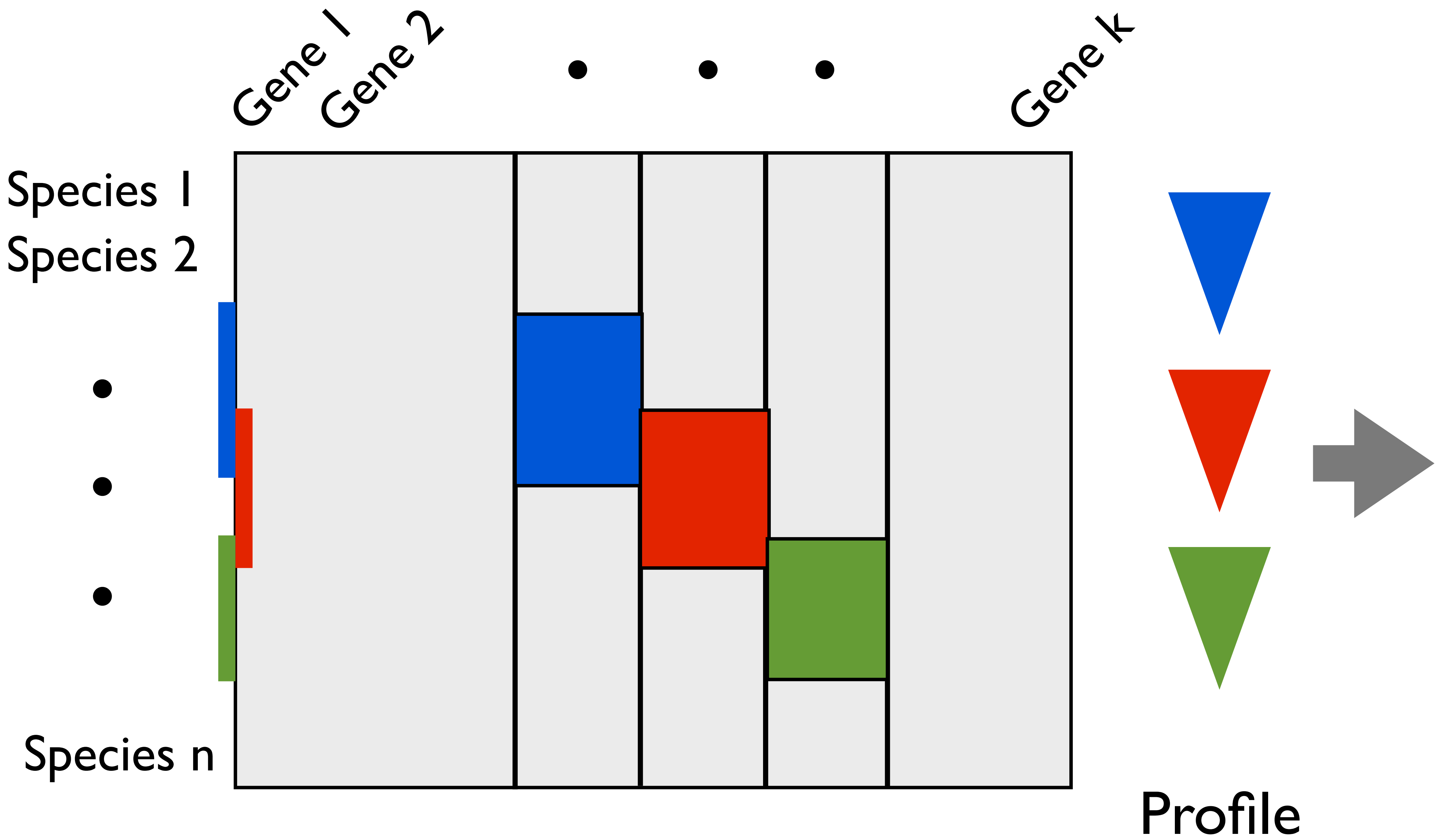

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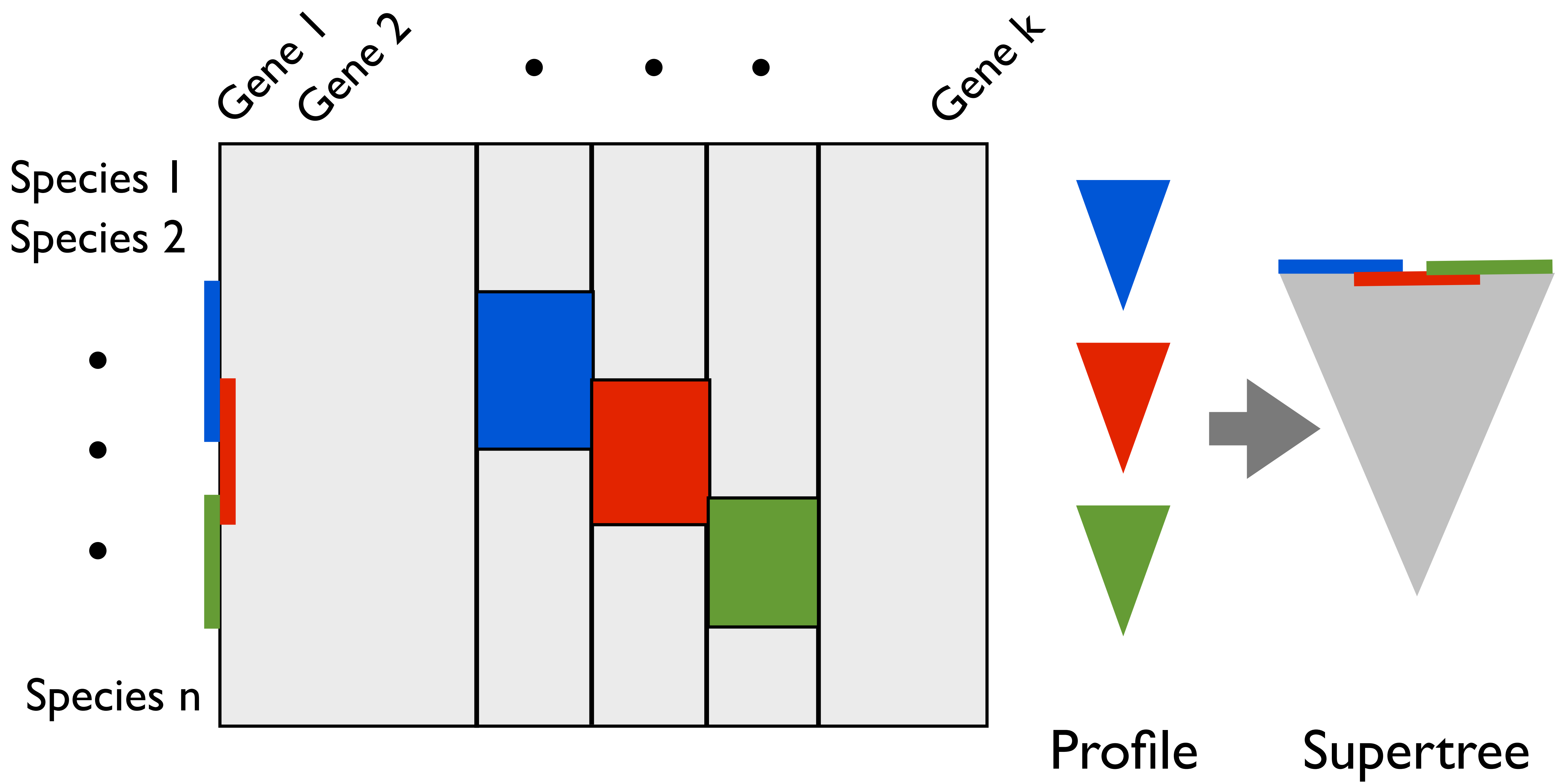

Gene 1
Gene 2
Gene k
Species 1
Species 2
Species n
Profile
Supertree

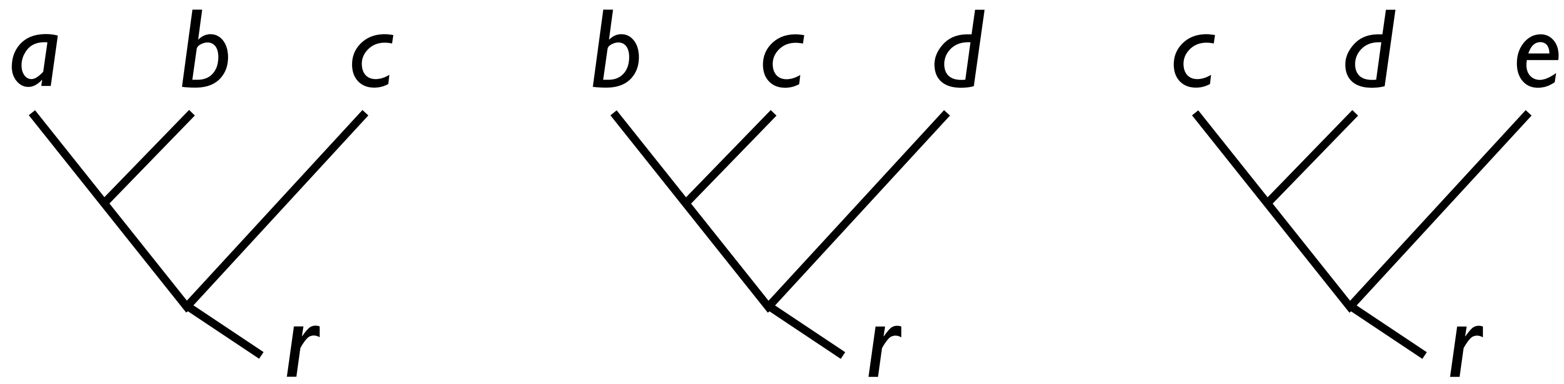
Agreement Supertrees

Agreement Supertrees

Input
Trees

Agreement Supertrees

Input Trees



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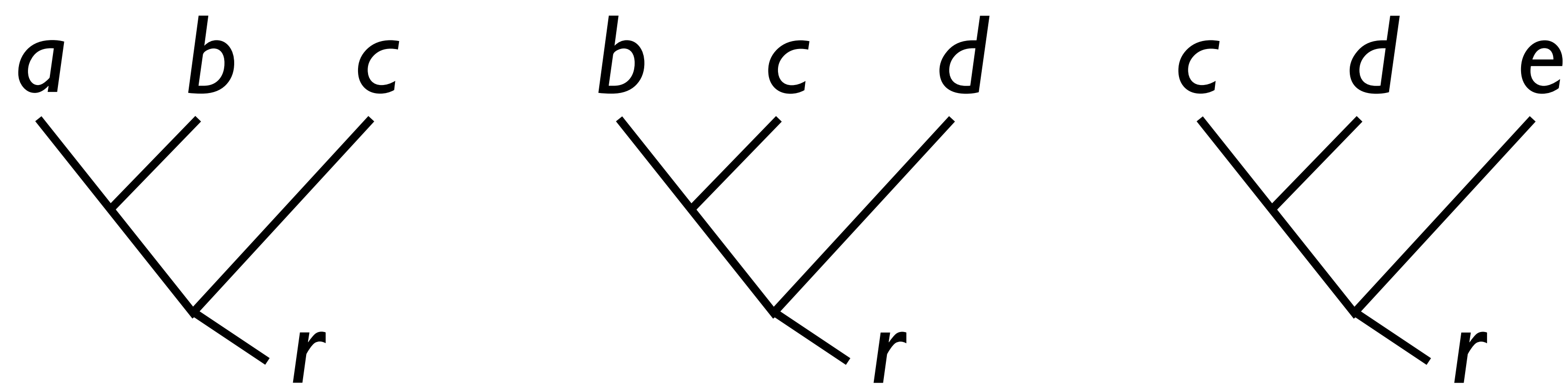

(Profile)

Agreement Supertrees

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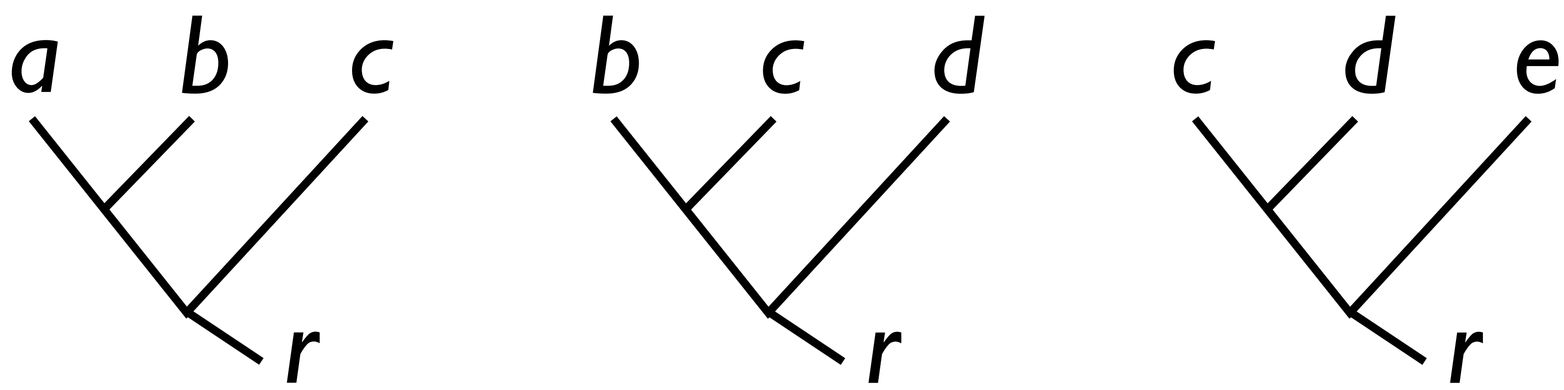

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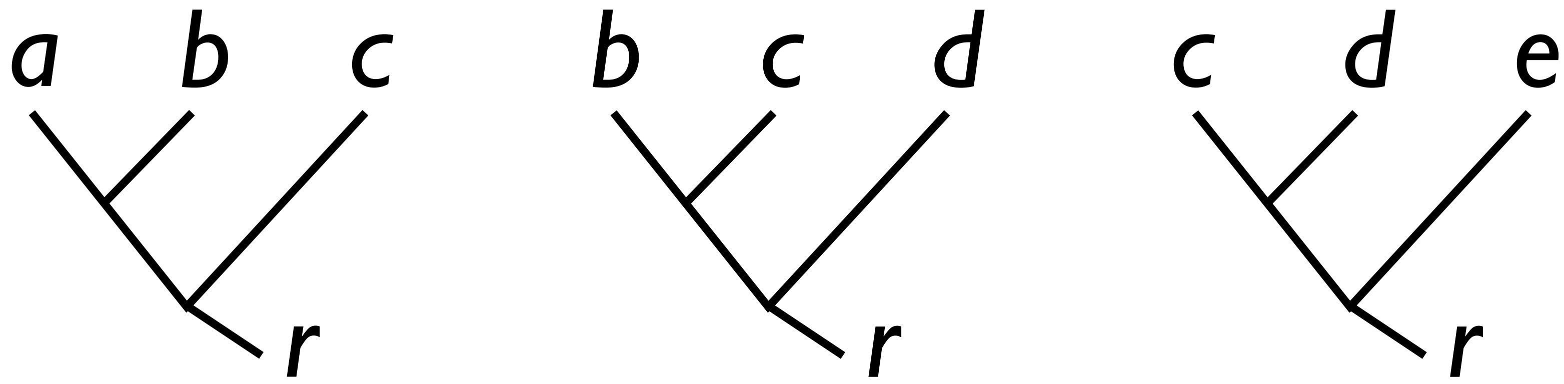

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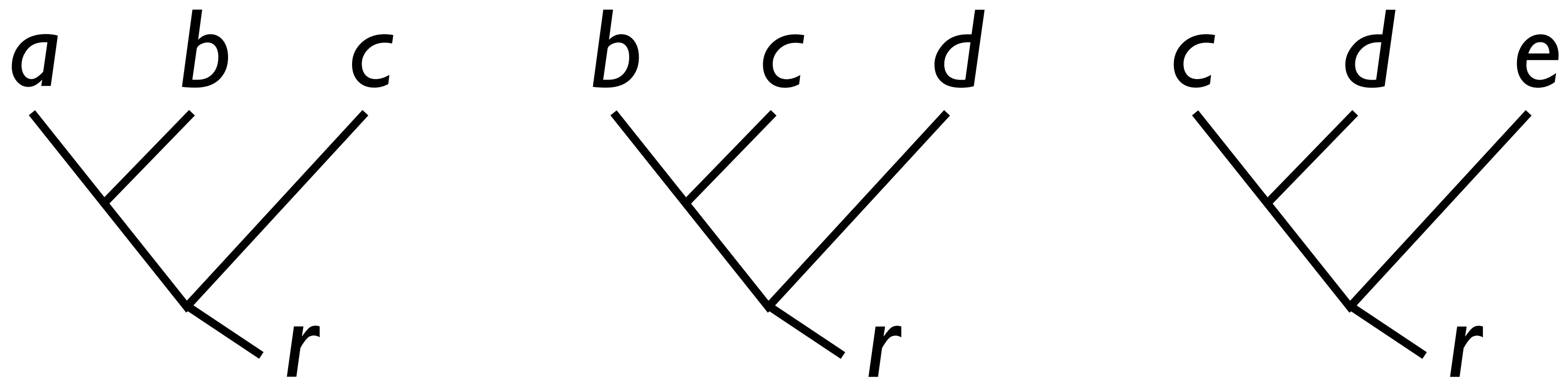

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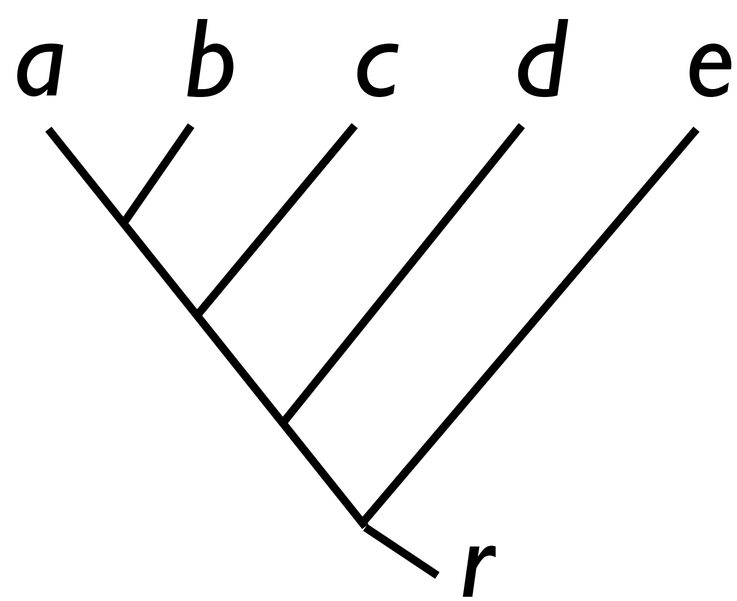

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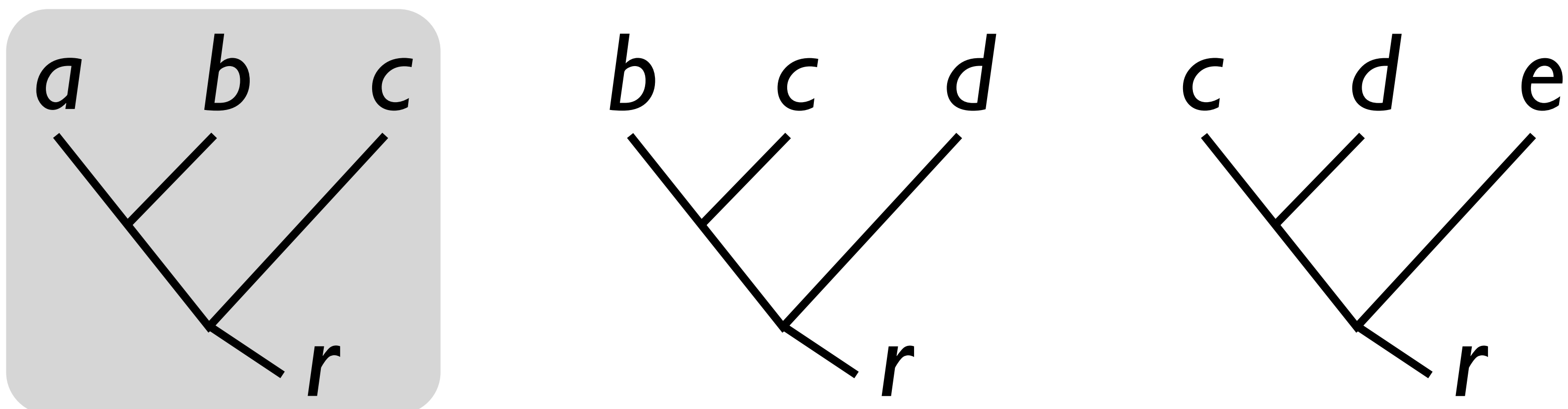

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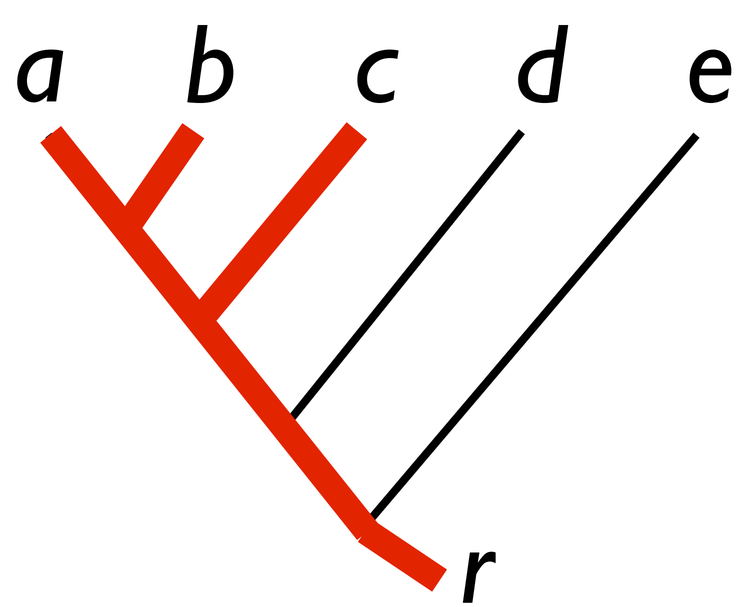

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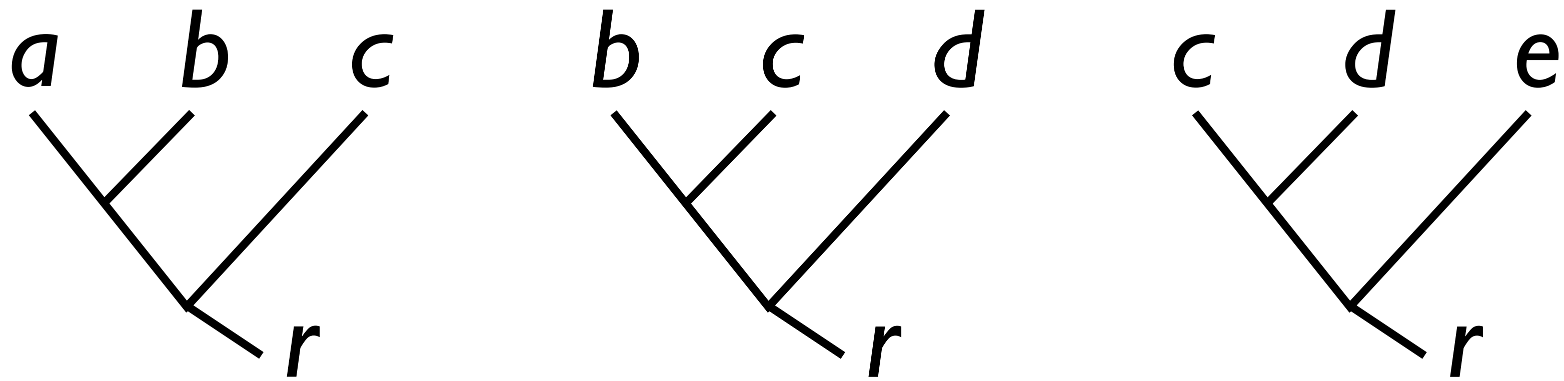

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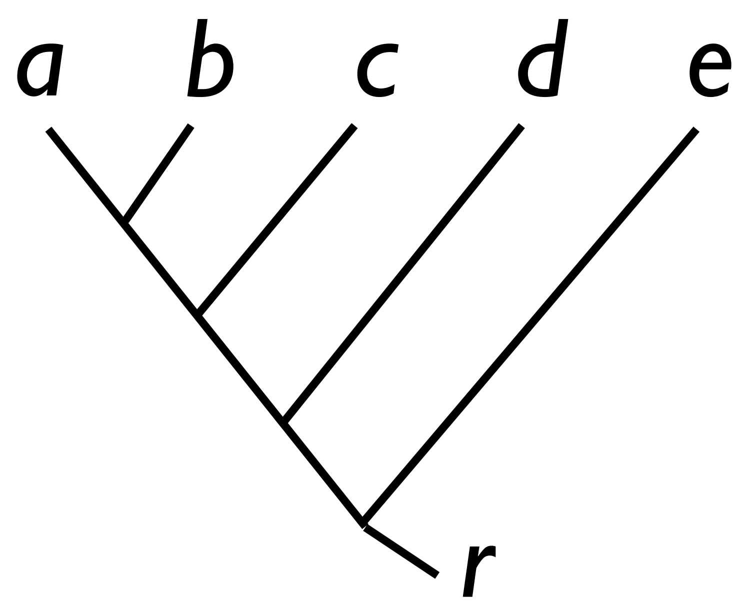

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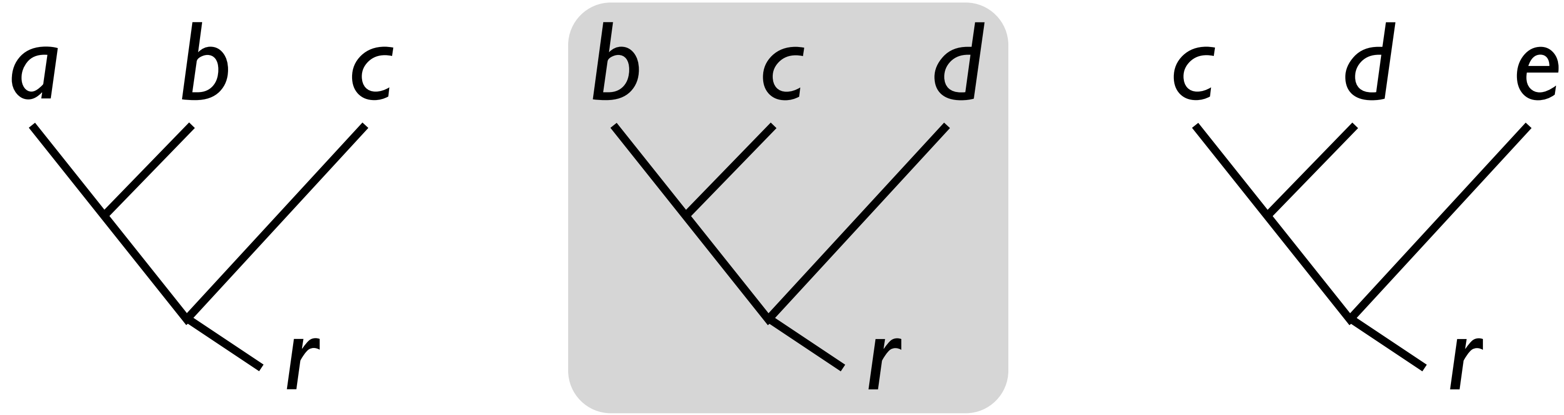

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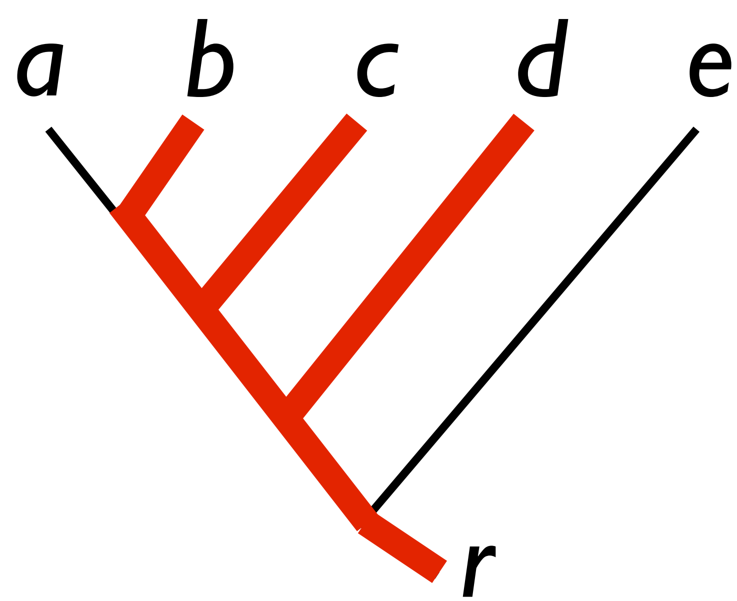

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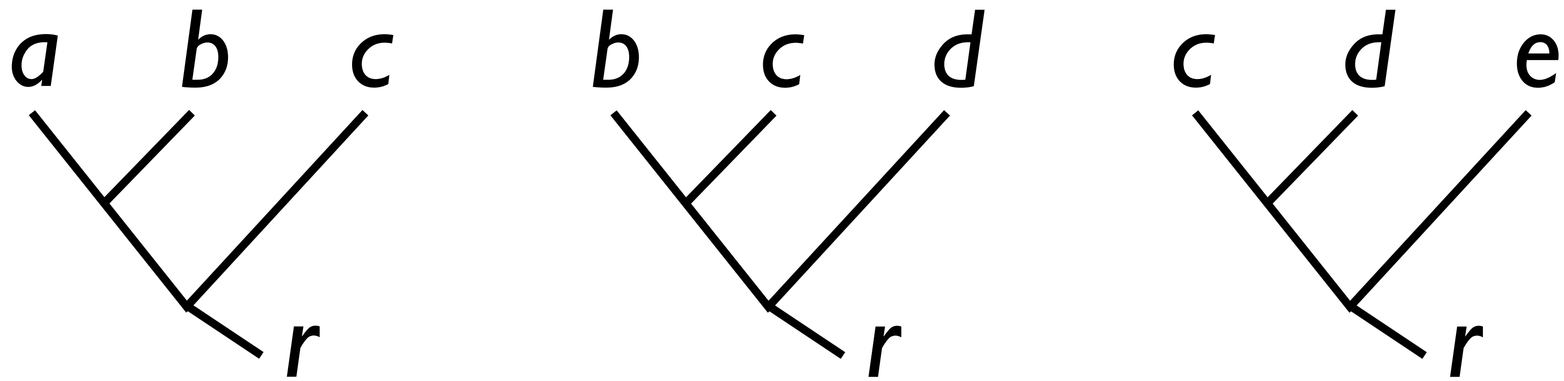

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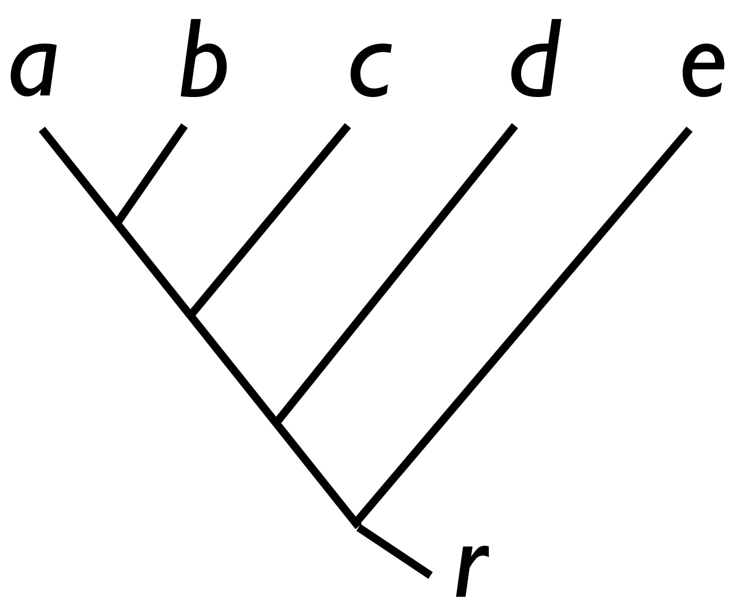

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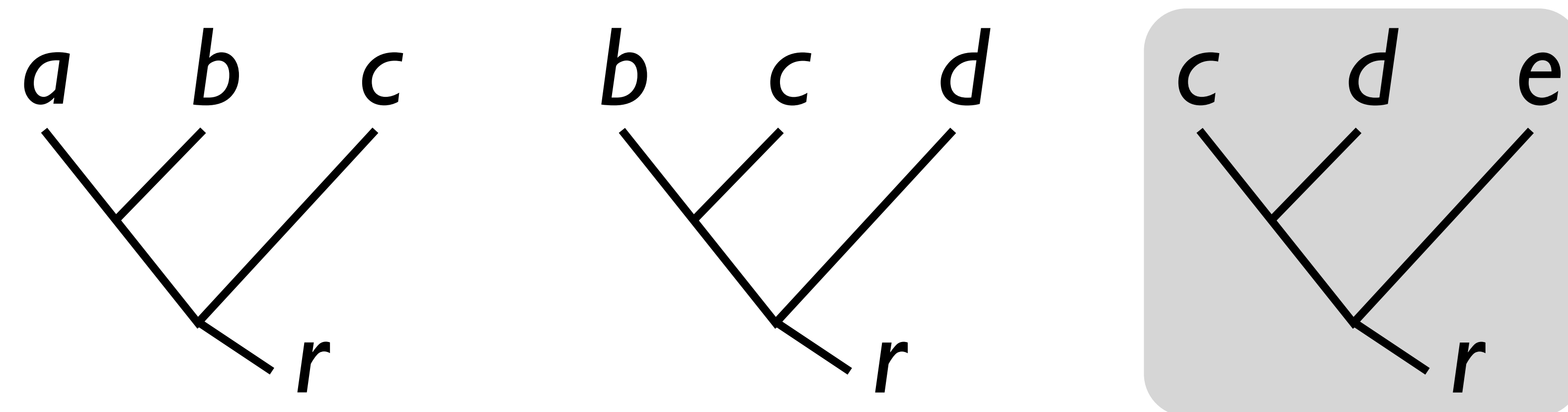

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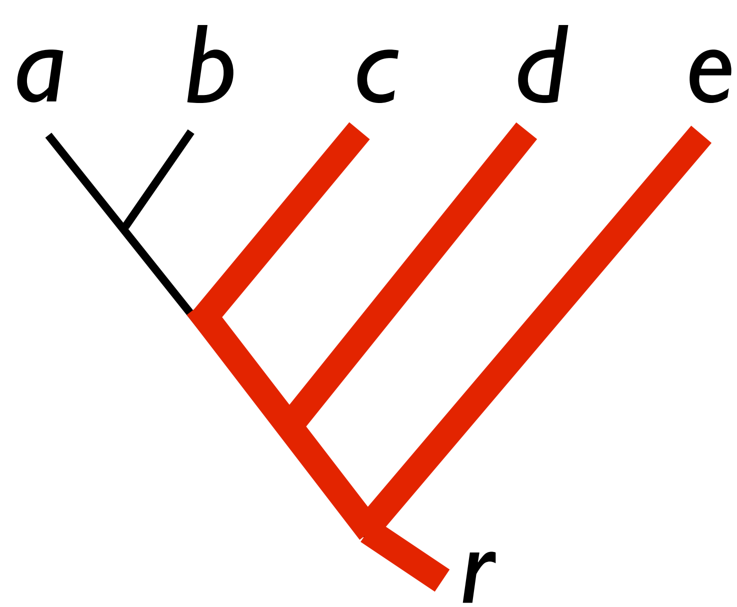

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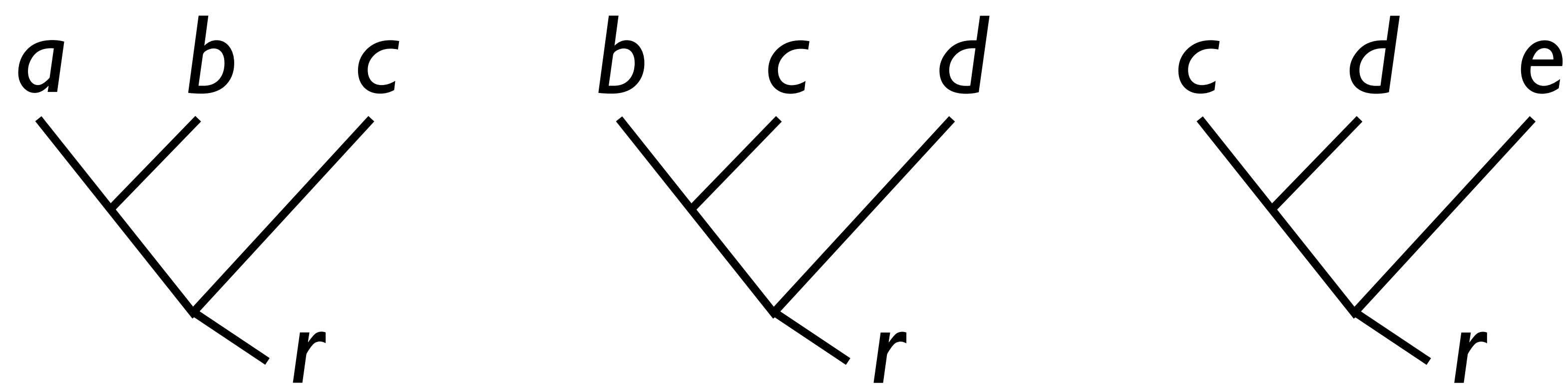

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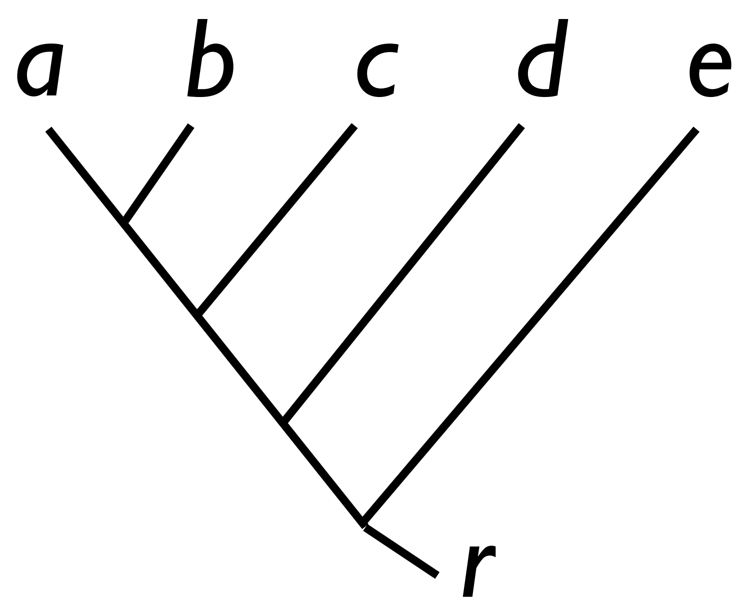

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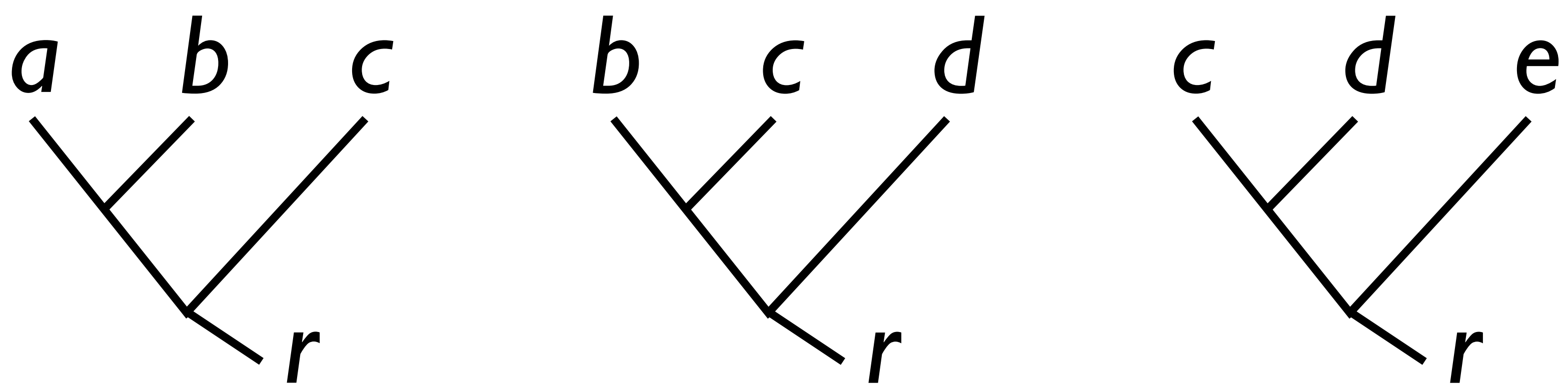

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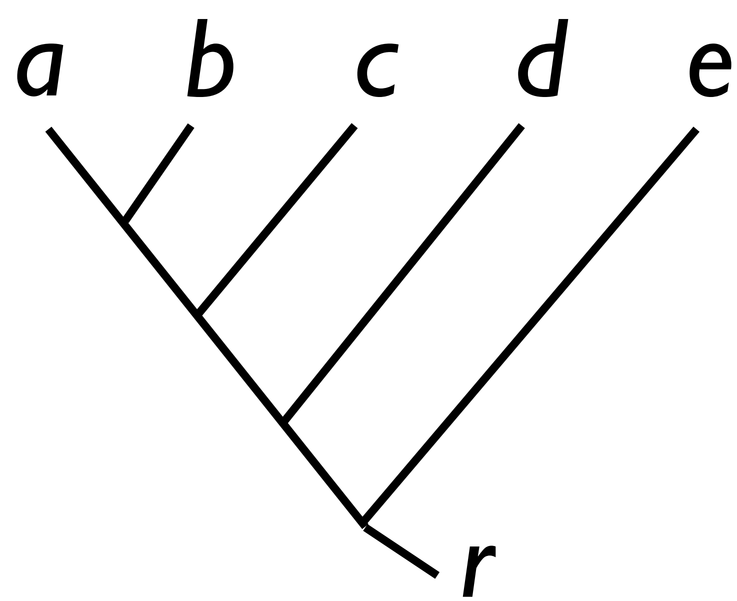


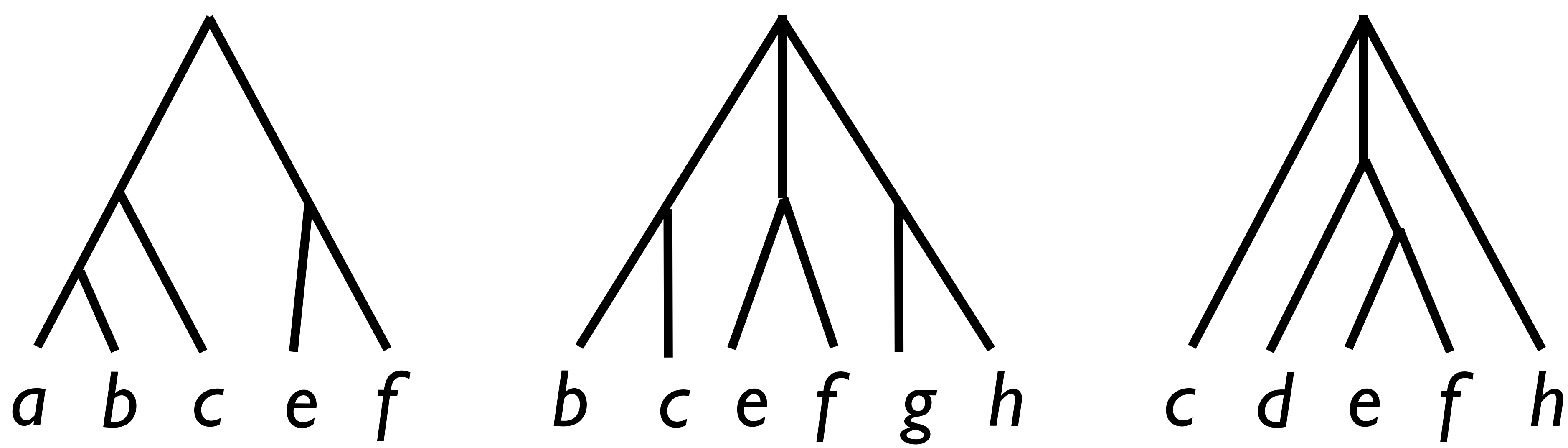
The input trees agree.

Agreement Supertrees: Definition

An agreement supertree for a profile $P = (T_1, T_2, \ldots, T_k)$ of rooted phylogenetic trees on n taxa is a tree S such that

- the leaf set of S is the union of the leaf sets of the trees in P and
- for each i, the restriction of S to the leaf set of T_i is T_i itself.

Agreement Supertrees



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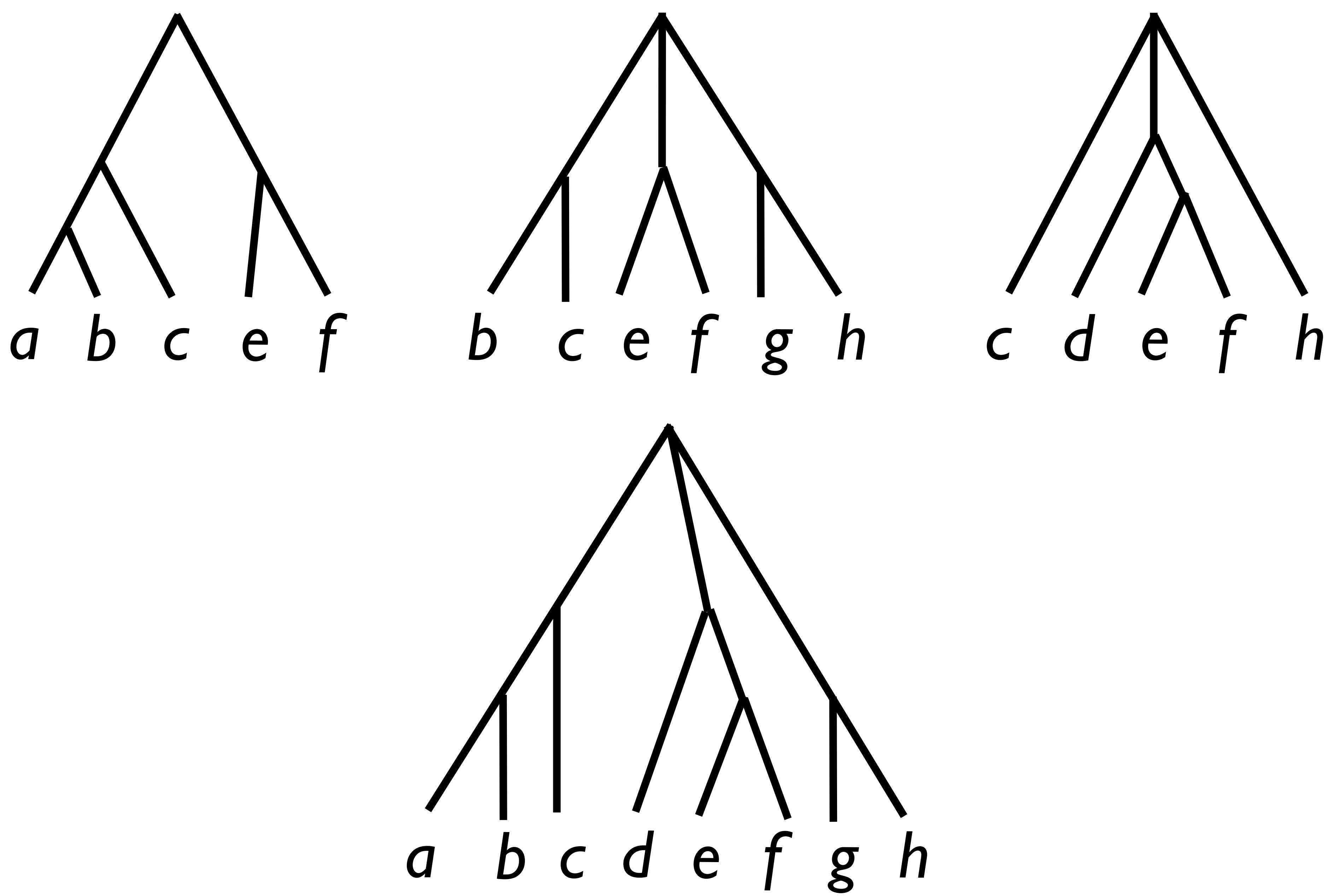

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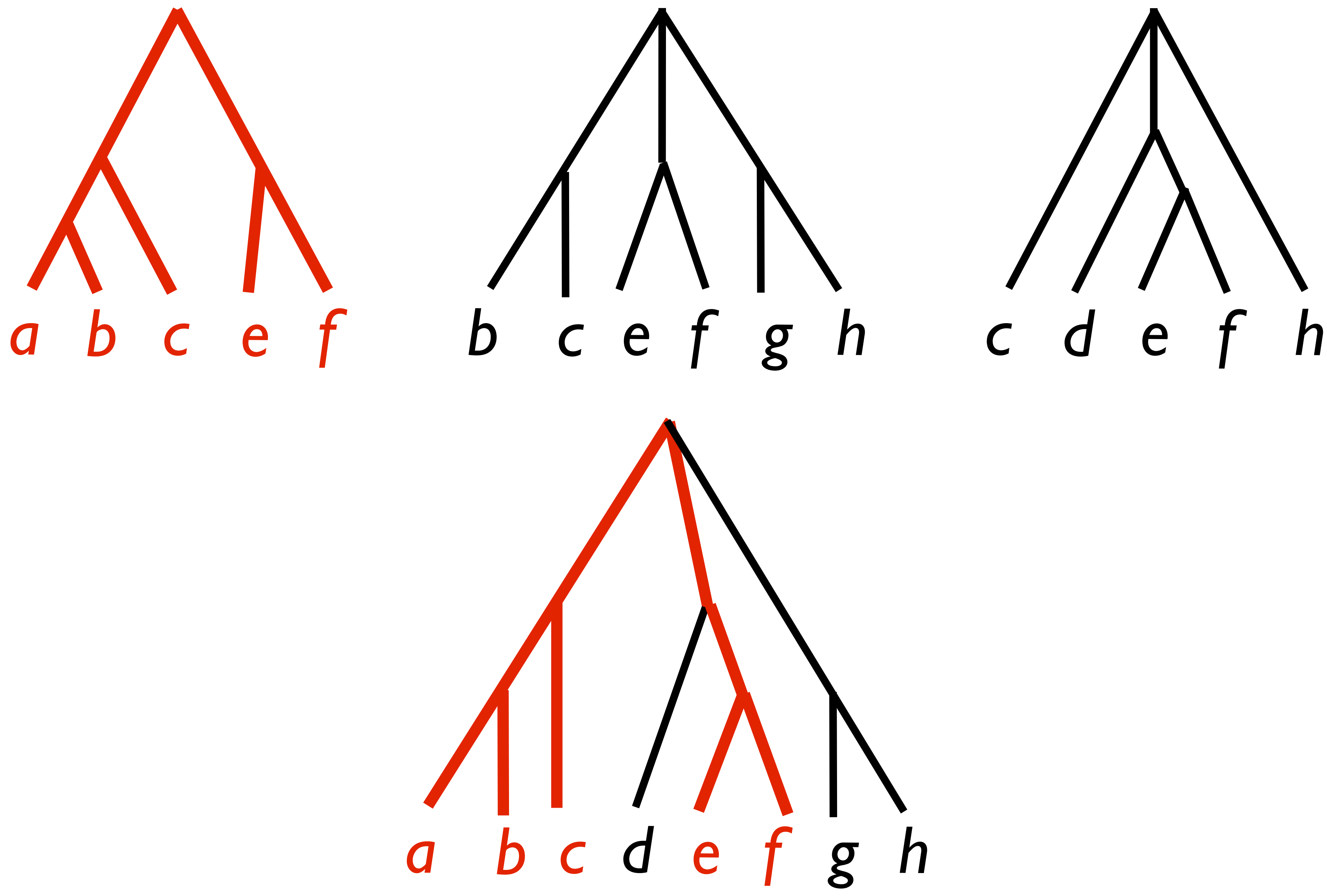

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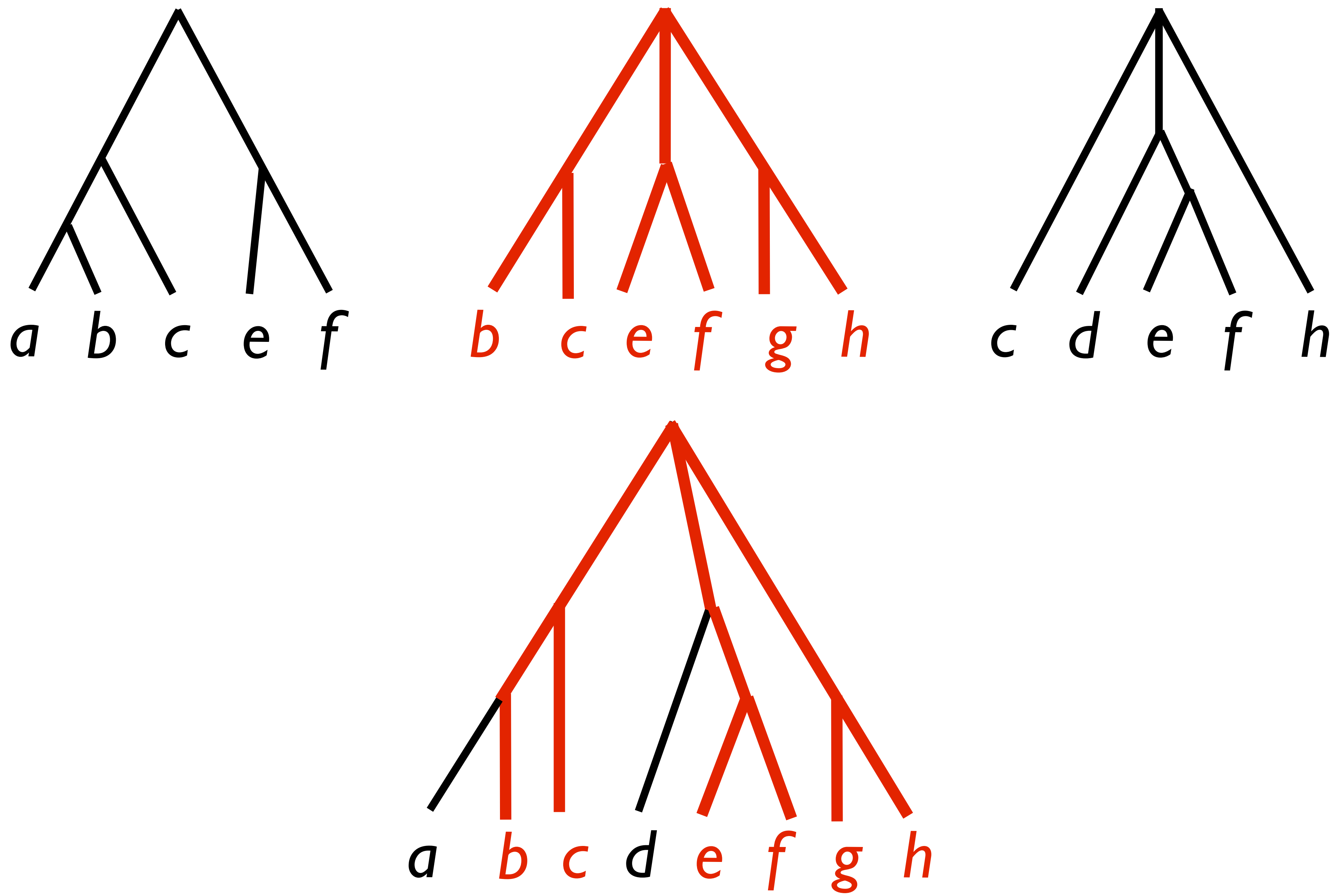

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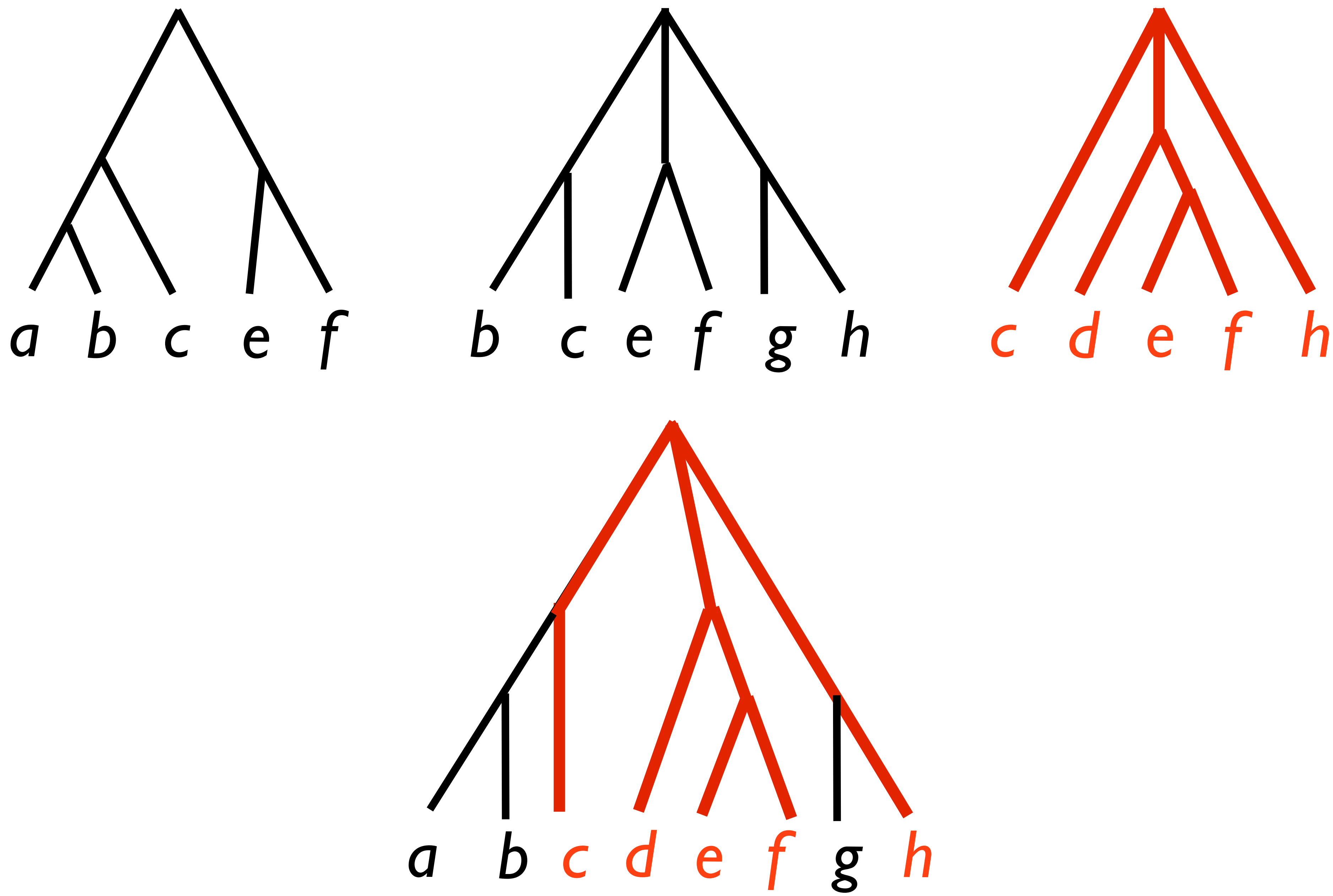

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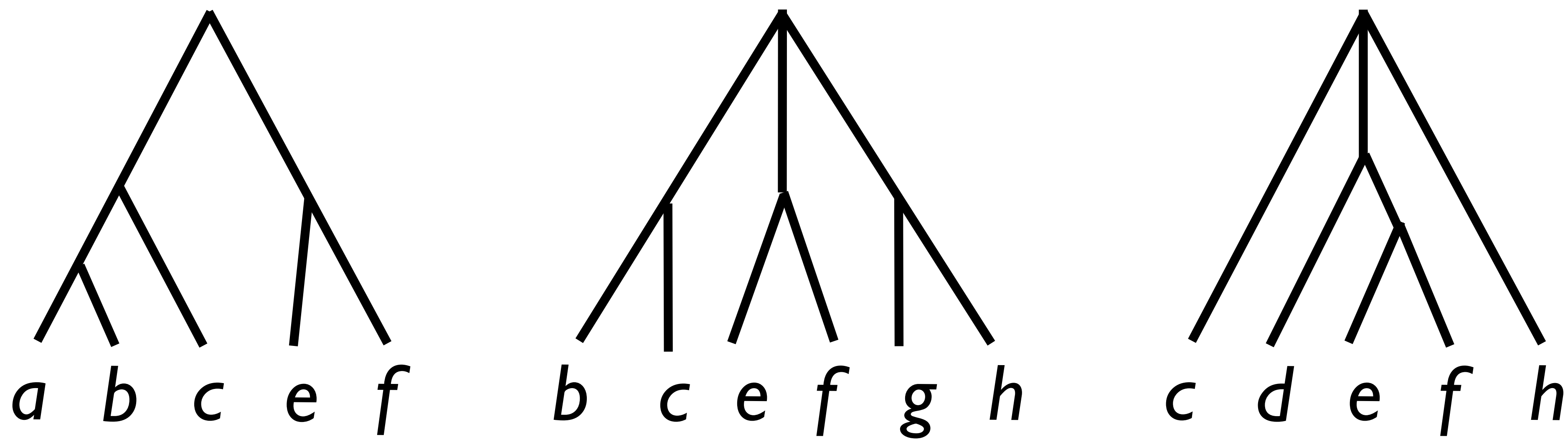

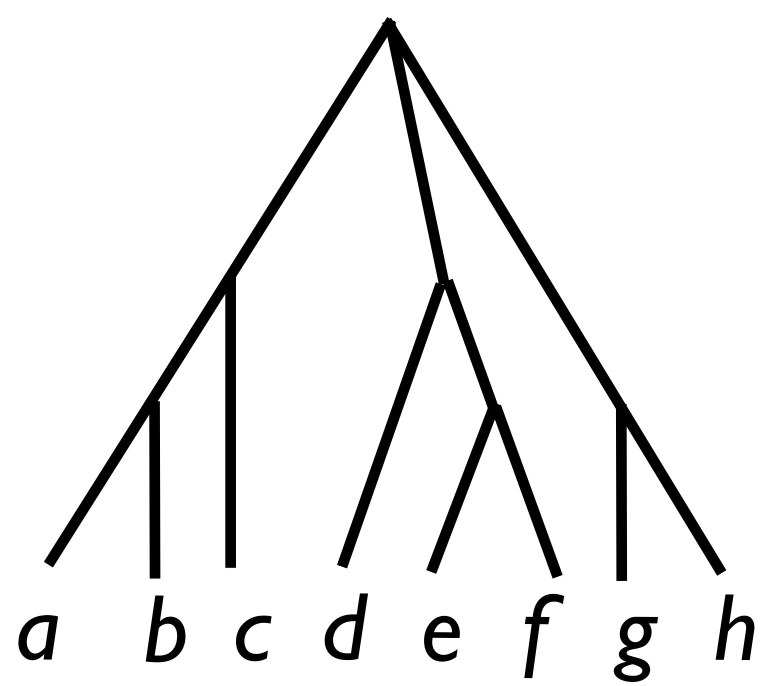

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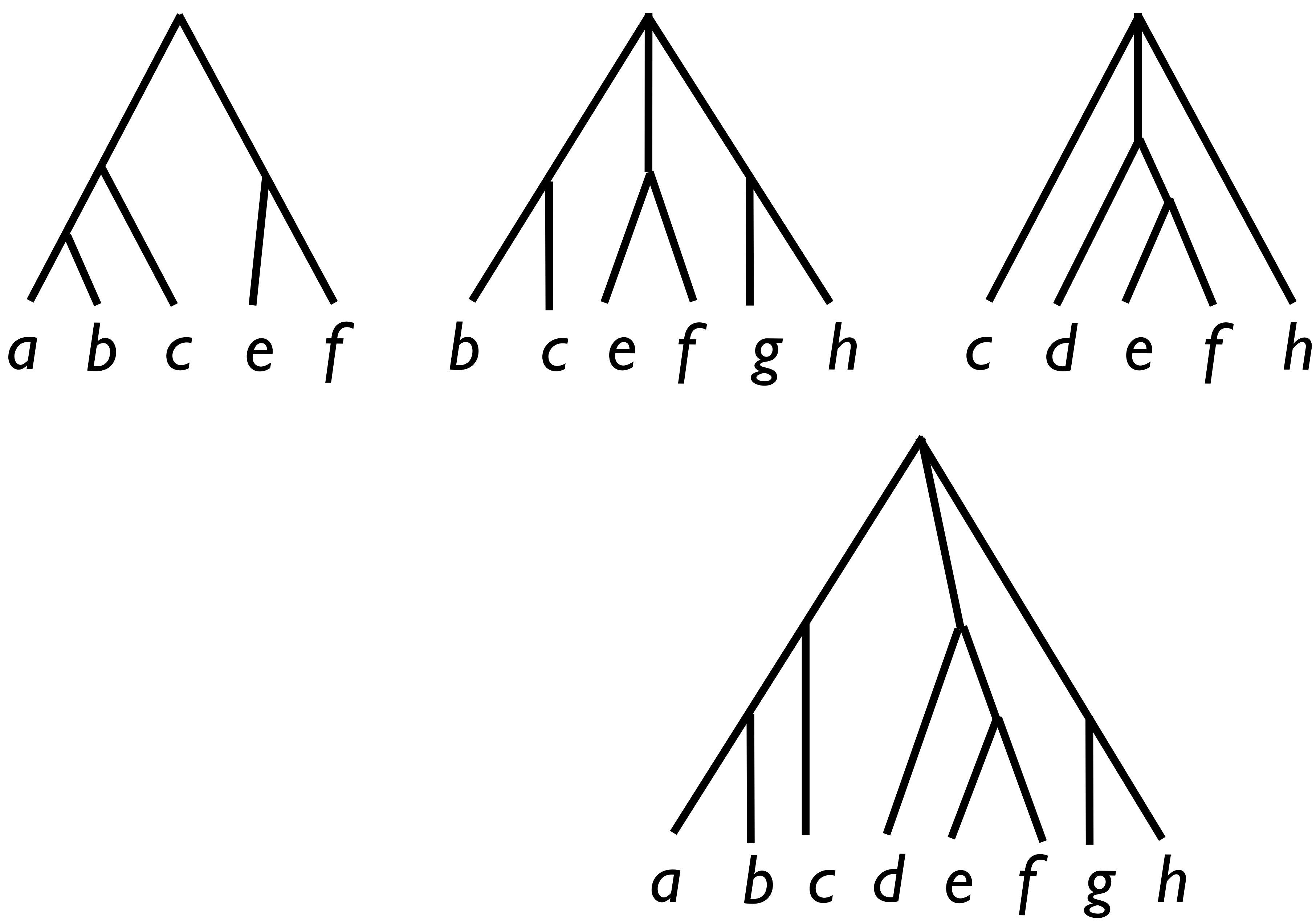

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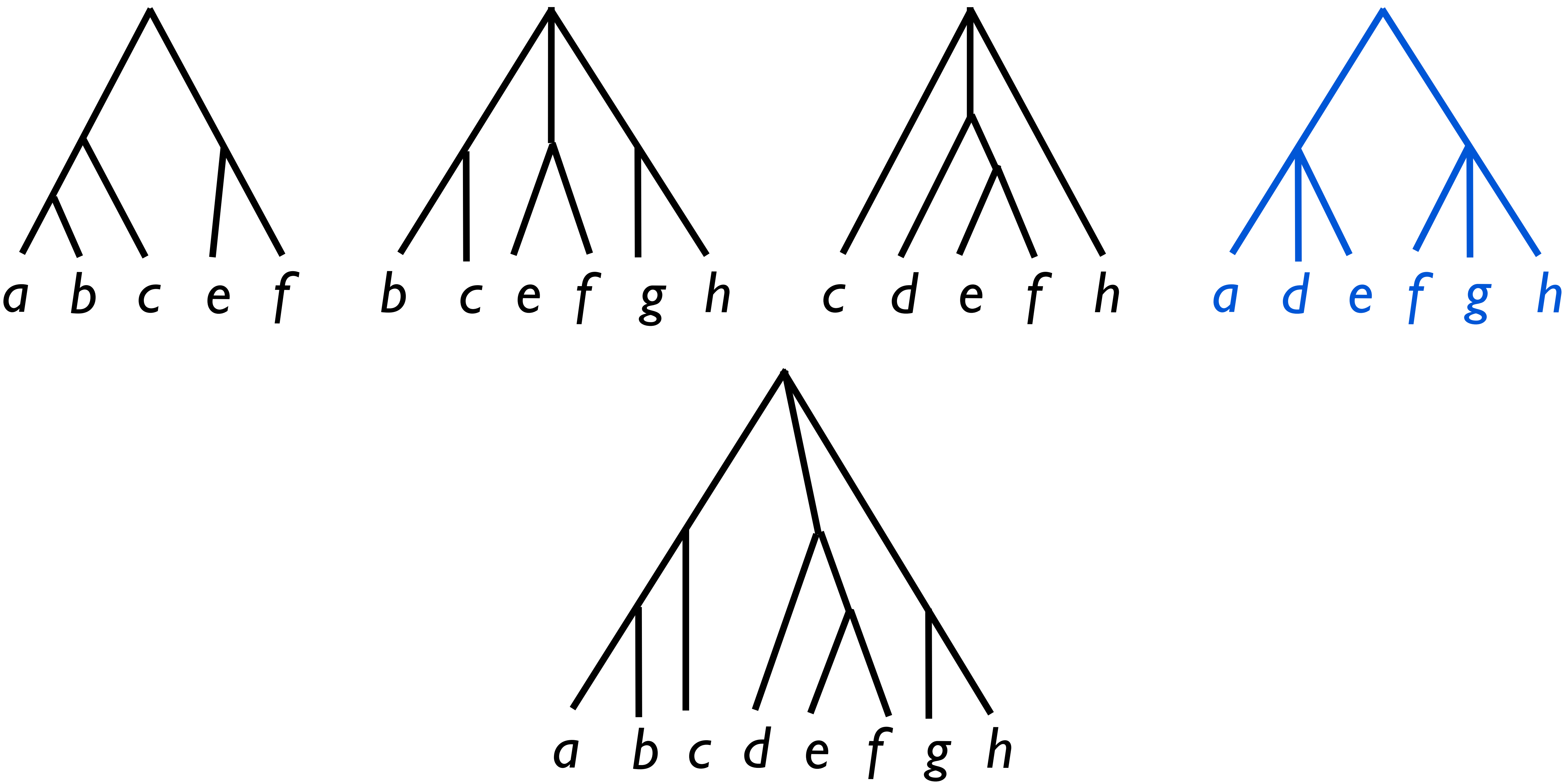

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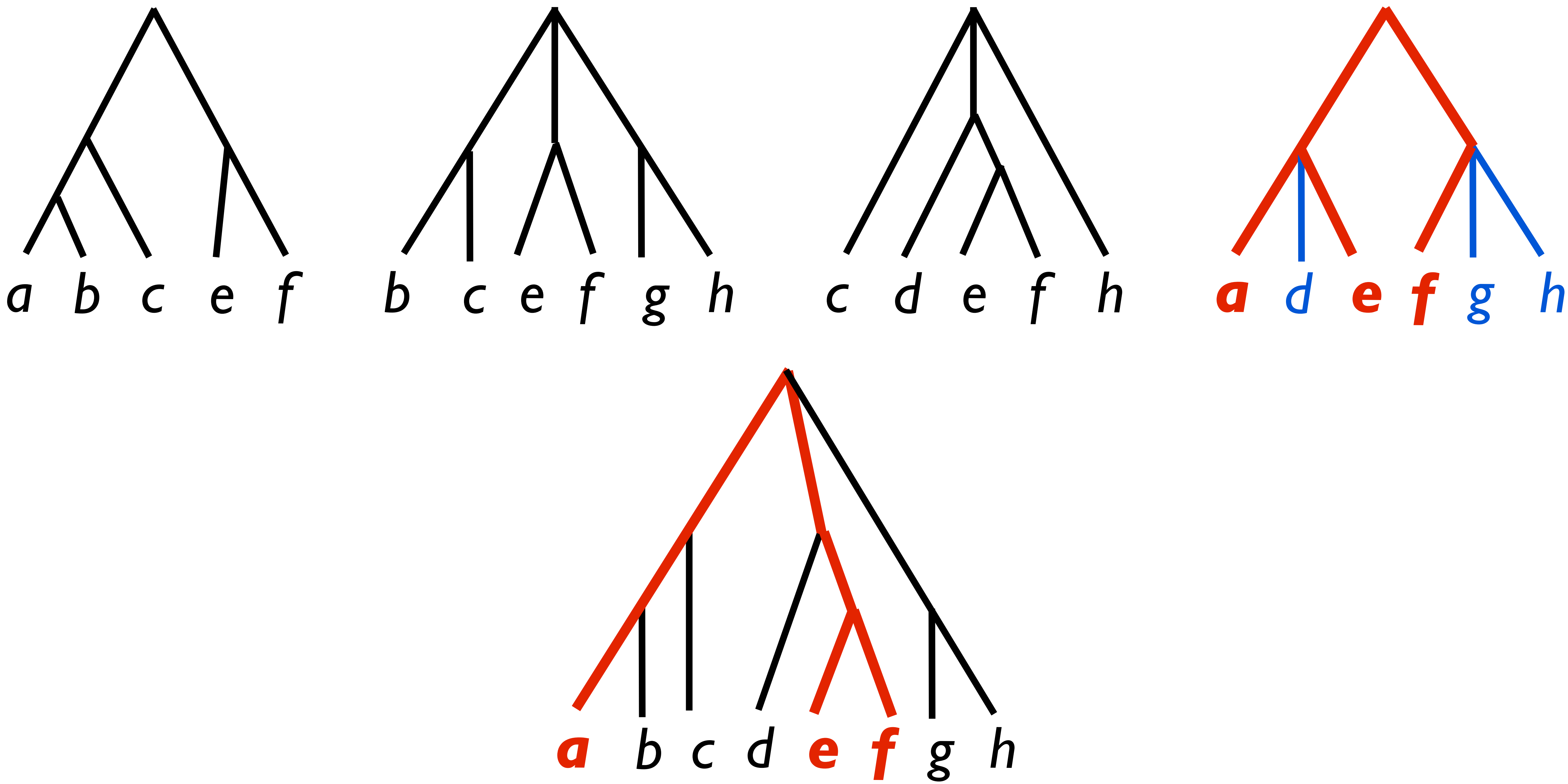

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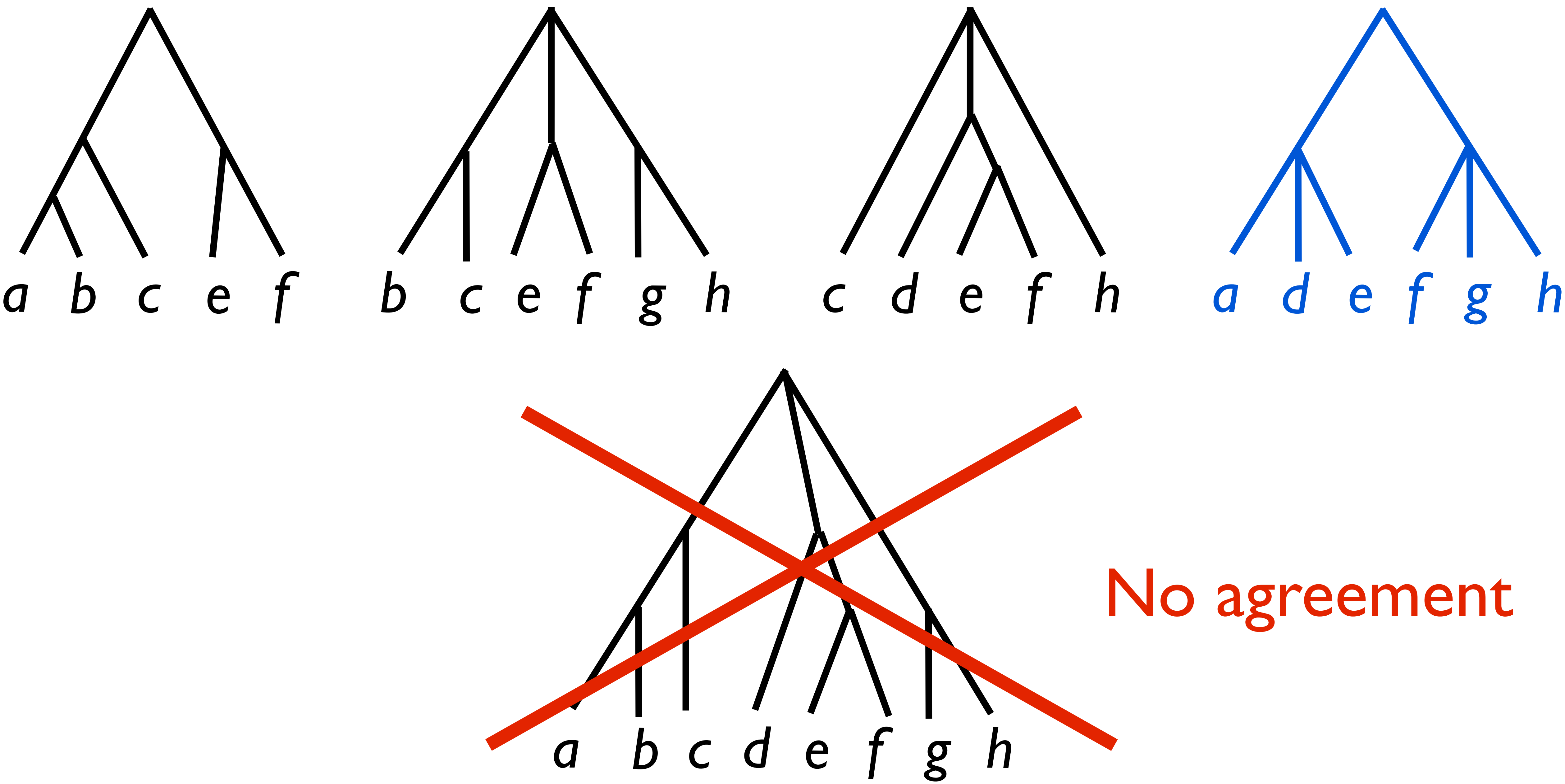

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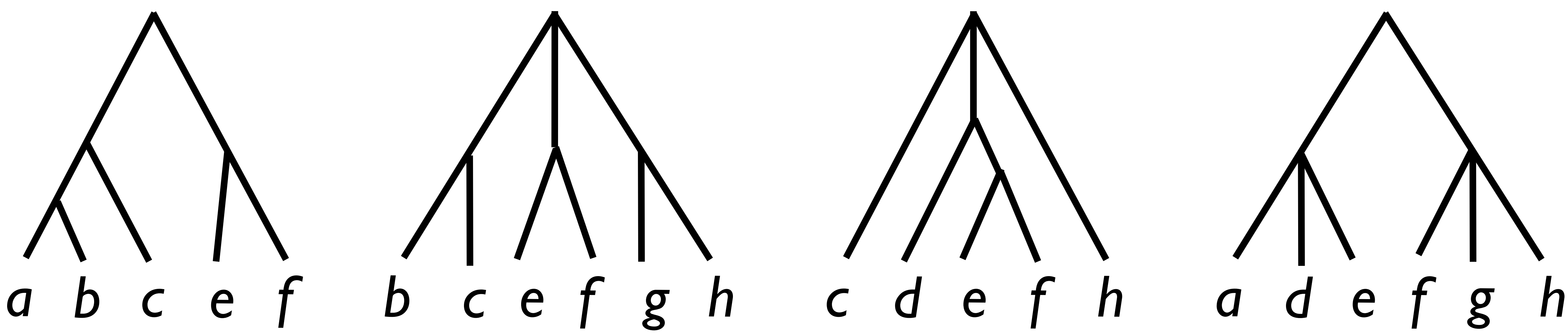
- **Taxon removal (AST-TR):** Can we remove q taxa from $P = (T_1, T_2, \ldots, T_k)$ so that P has an agreement supertree?

Taxon Removal



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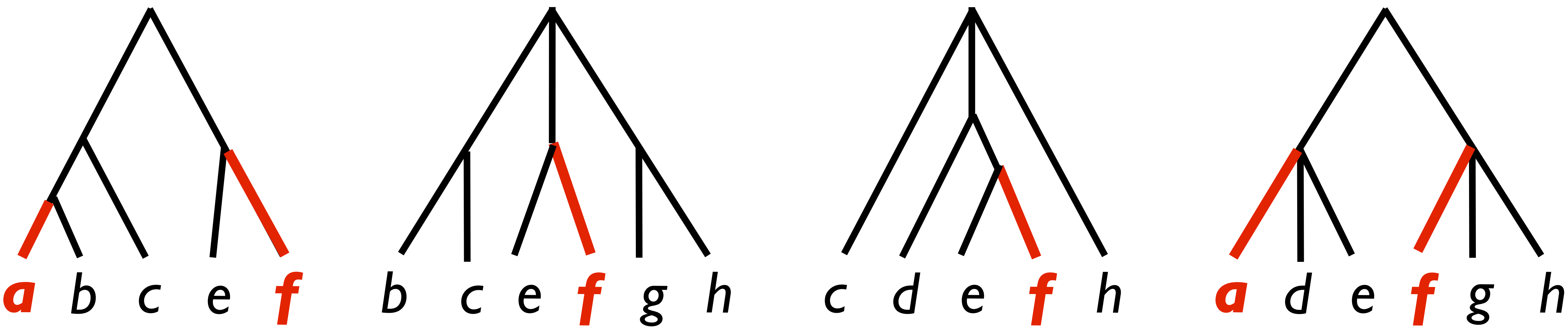

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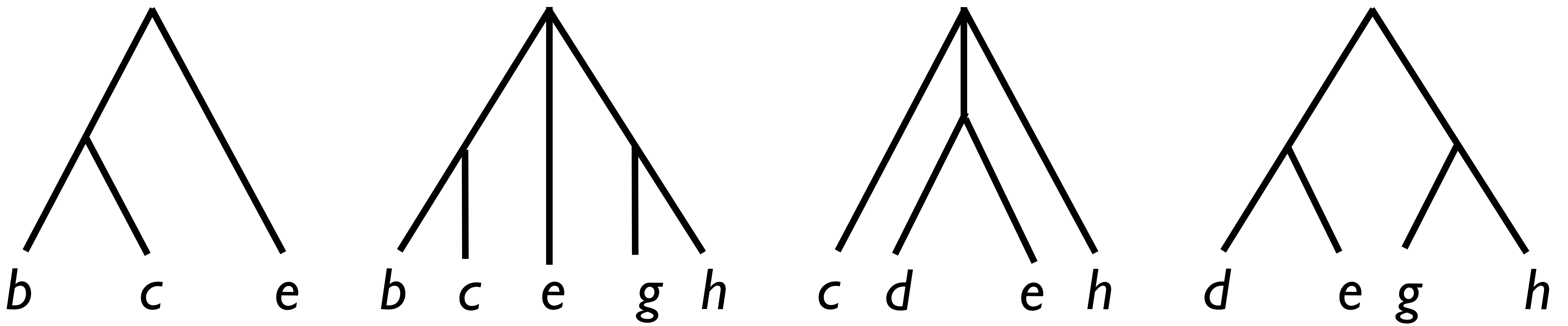

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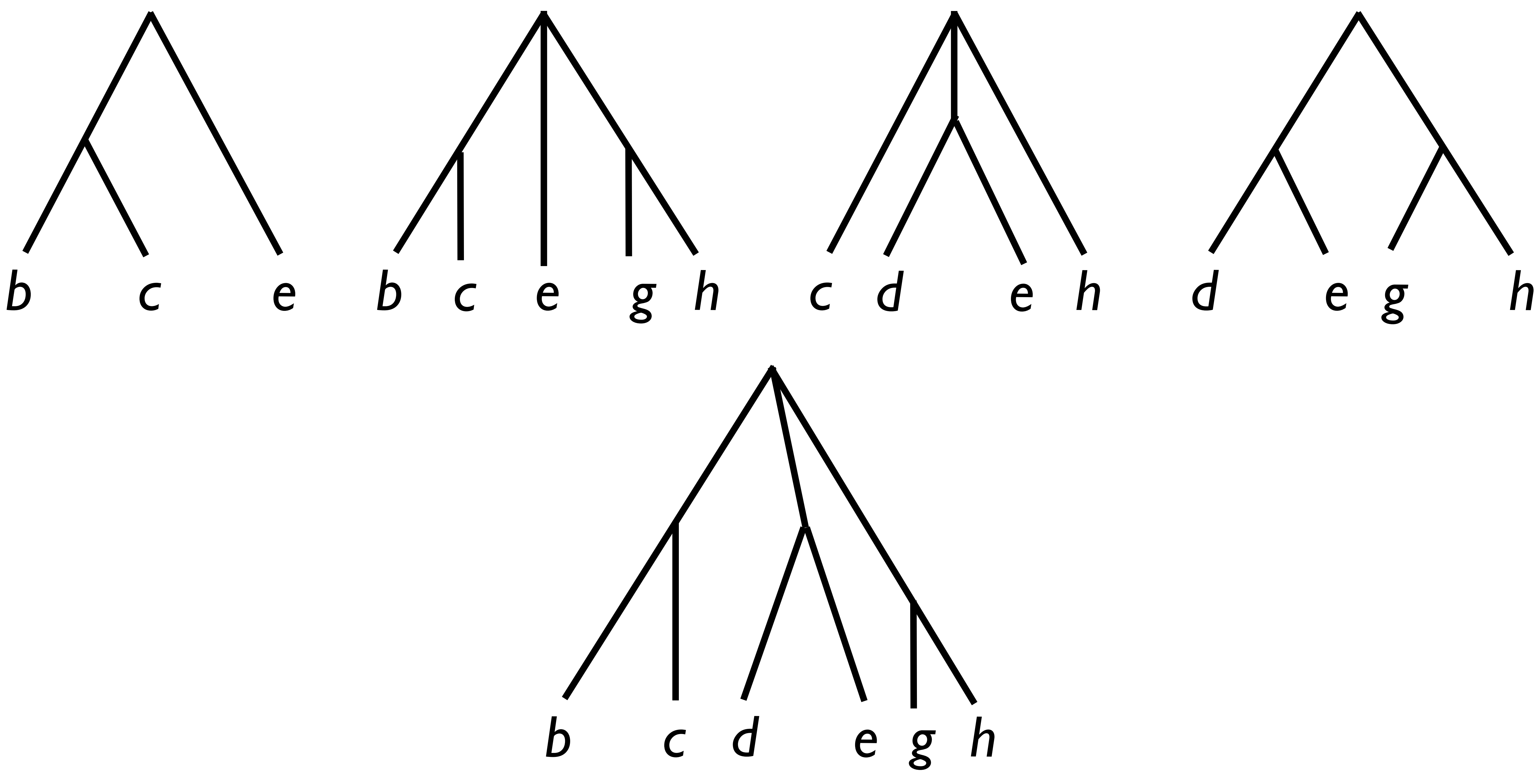

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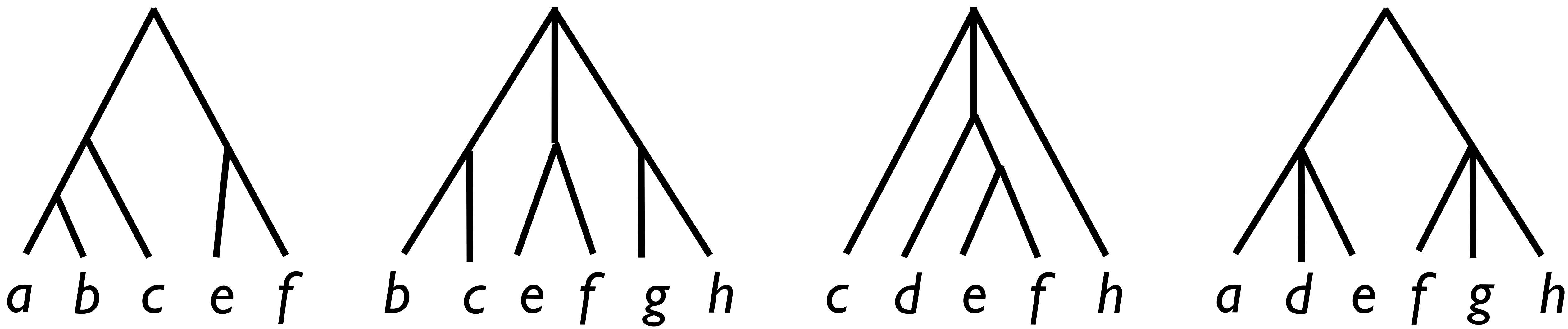

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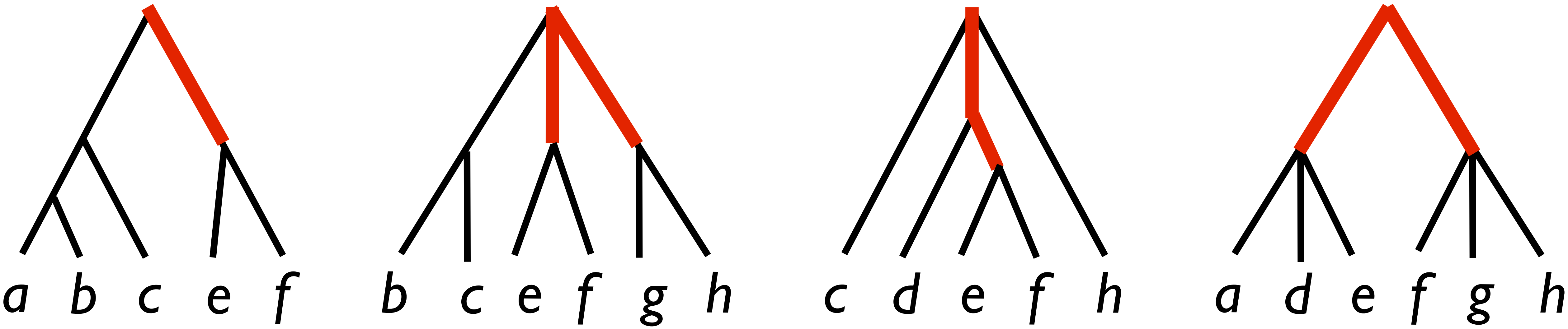

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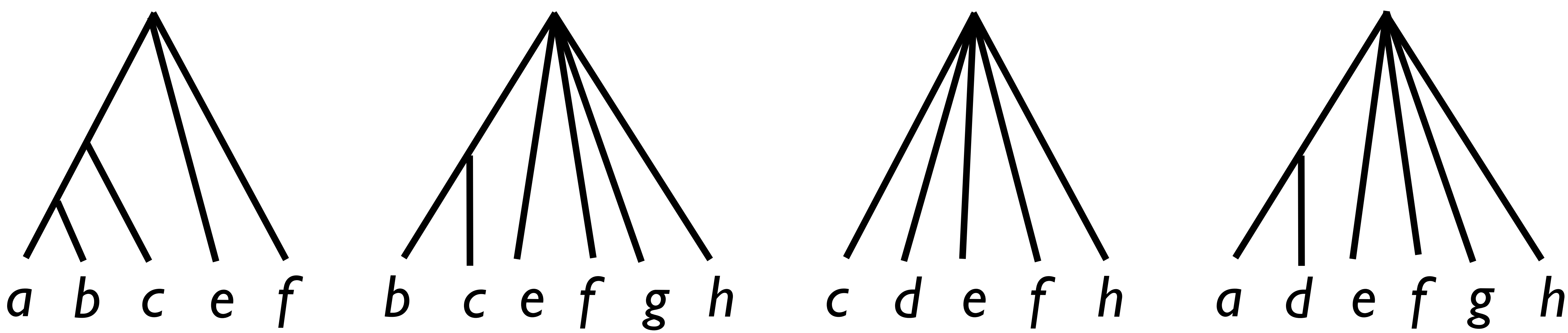

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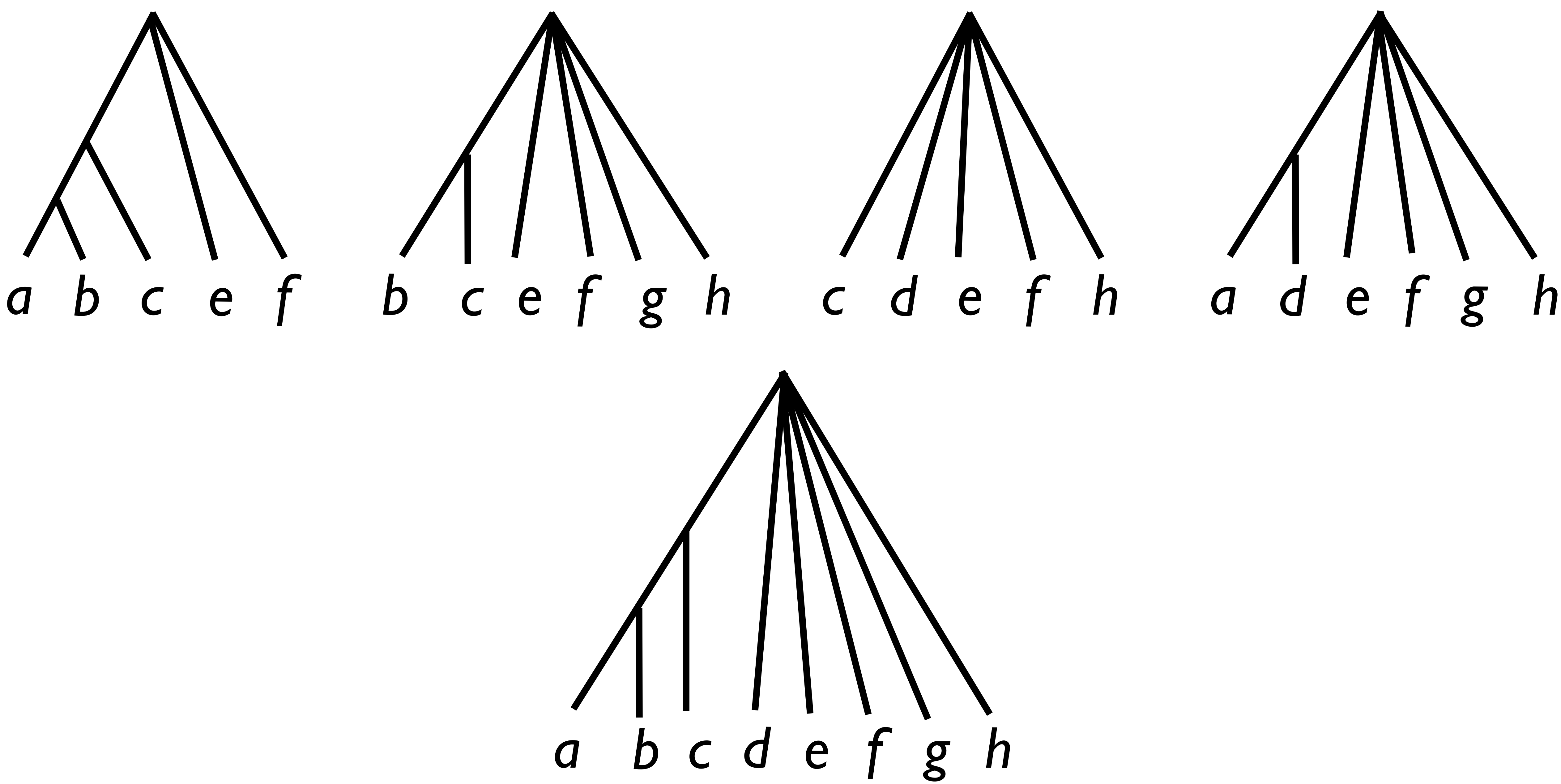

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- AST-TR and AST-EC with input trees of arbitrary degree are fixed-parameter tractable in k and q.
  - $O((2k)^p \, kn^2)$-time algorithms.

Previous Work: AST-TR

- NP-hard (Jansson et al., 2005, Berry & Nicolas 2007).
- Fixed-parameter tractable in k and q for binary input trees (Guillemot & Berry 2010).
- Fixed-parameter intractable when parameterized by only k or q (Berry & Nicolas 2007)
- Maximum Agreement Supertree: Find largest subset of taxa on which trees agree (Berry & Nicolas 2007, Jansson et al. 2005, Kao 2007).
    - Dual of optimization version of AST-TR.
    - Exact algorithms:
        - Binary: $O(6^k n^k)$ (Guillemot & Berry 2010, Hoang & Sung 2011).
        - Maximum degree d: $O((kd)^{kd+3} 2^k n^k)$ (Hoang & Sung 2011).

Complexity of AST-EC

AST-EC is NP-complete

Reduction from Multicut:

Multicut

Input: $G = (V, E)$, $R \subseteq V \times V$, and integer q.

Question: Does there exist $S \subseteq E$, with $|S| \leq q$ s.t. for every $uv \in R$, u and v are in different components of $G \setminus S$?

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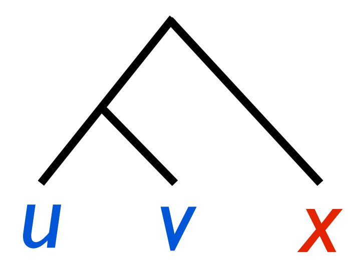

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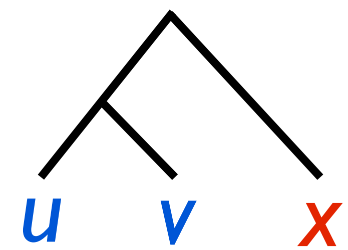
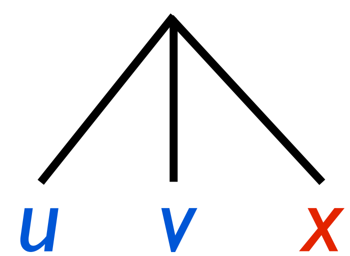


(ii) For every $uv \in R$:



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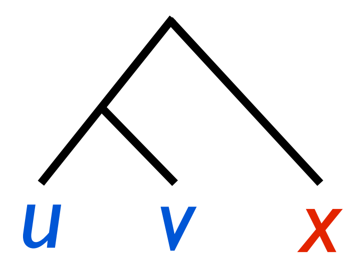

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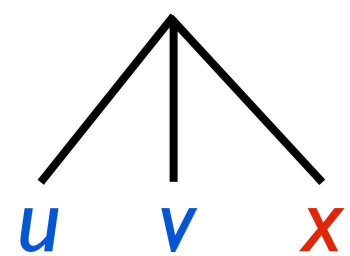
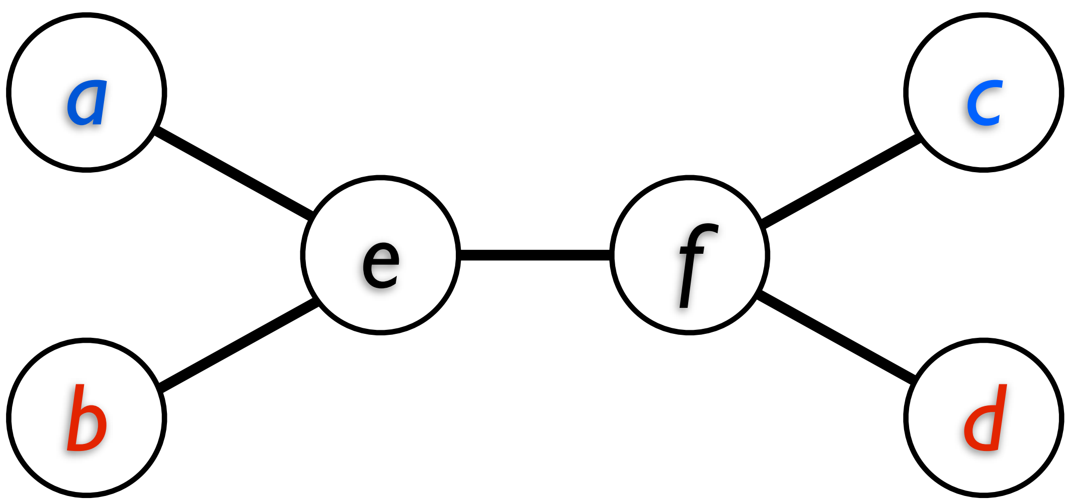


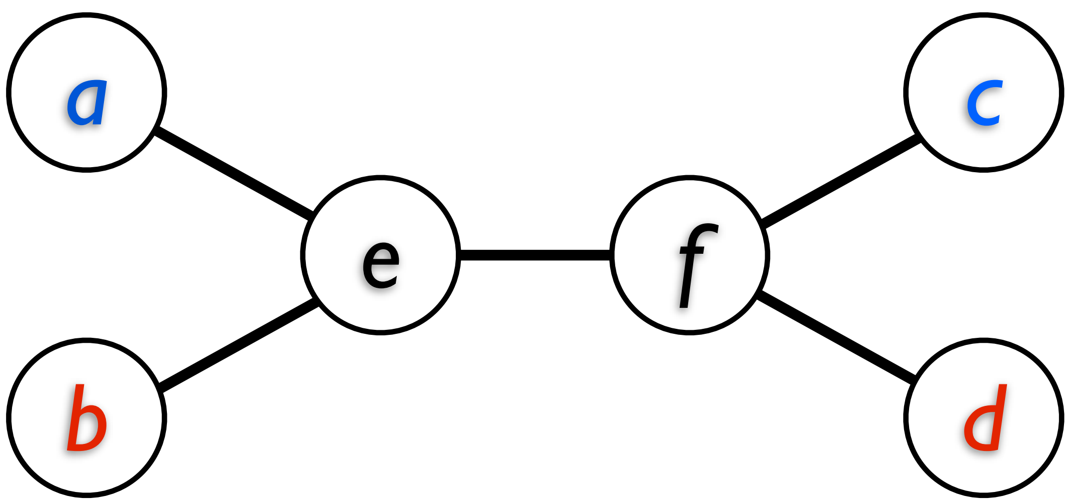
Claim. G has a cut of size $\leq q$, iff (P, q) has a "yes" answer

Reduction from Multicut



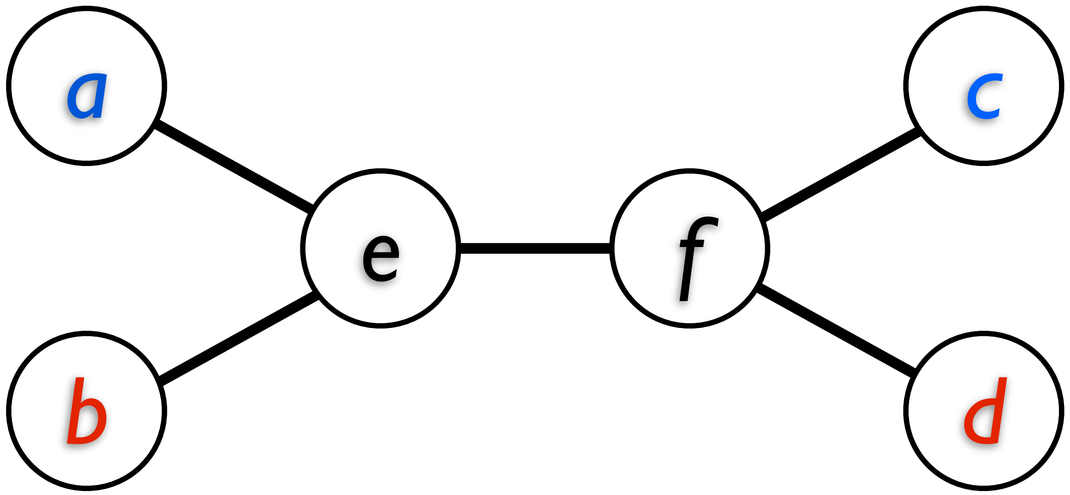
$R = \{ac, bd\}$

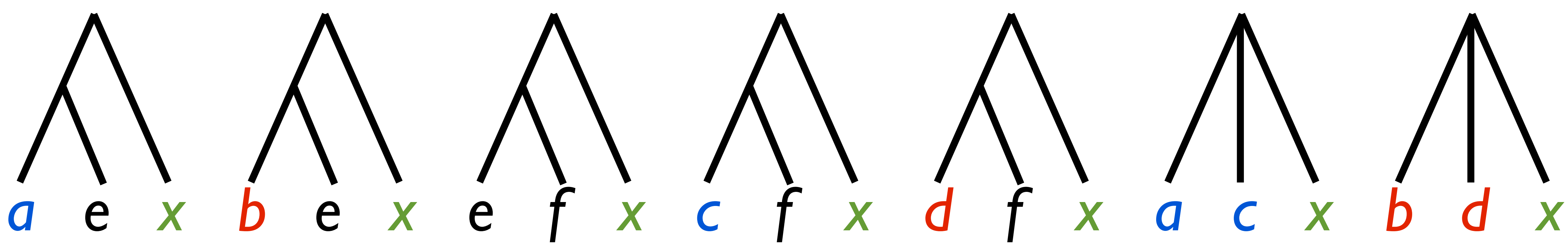
Reduction from Multicut



Edges

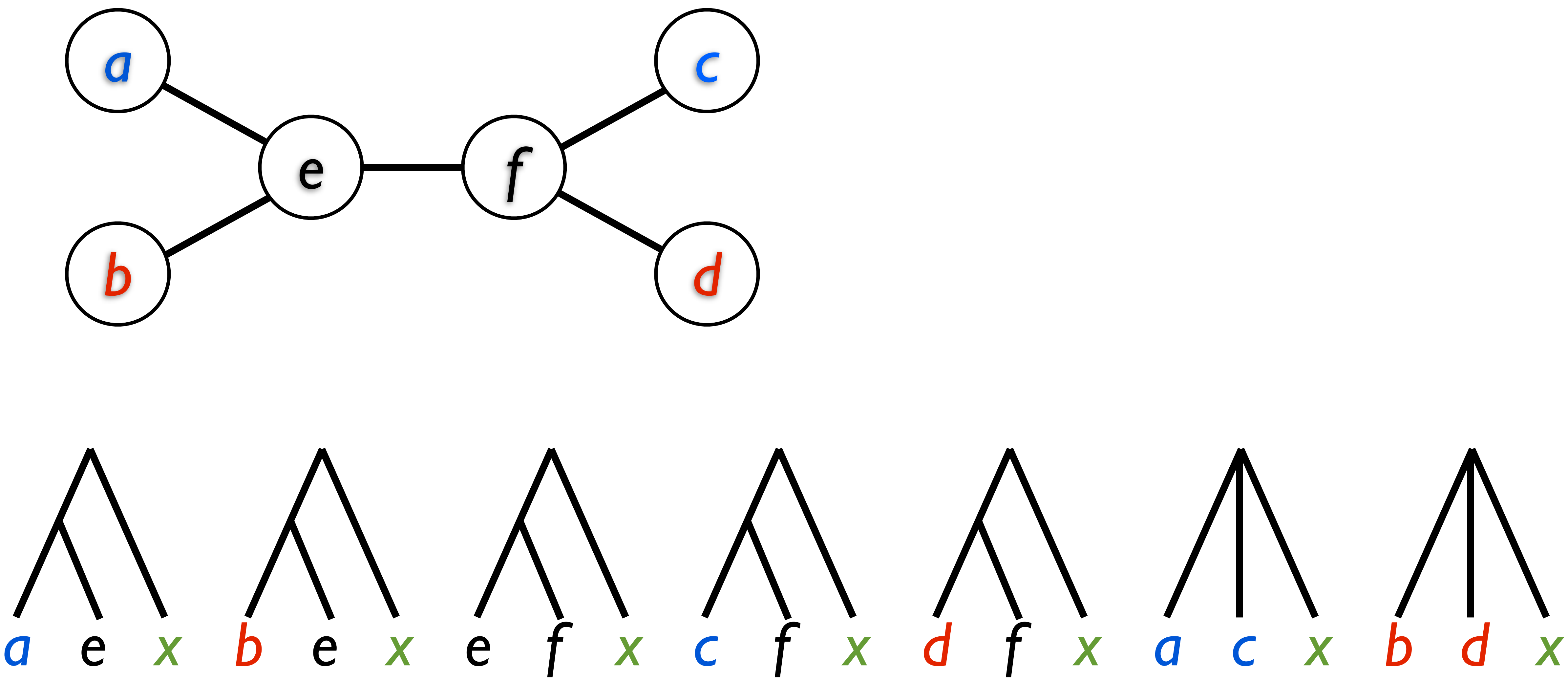
Reduction from Multicut



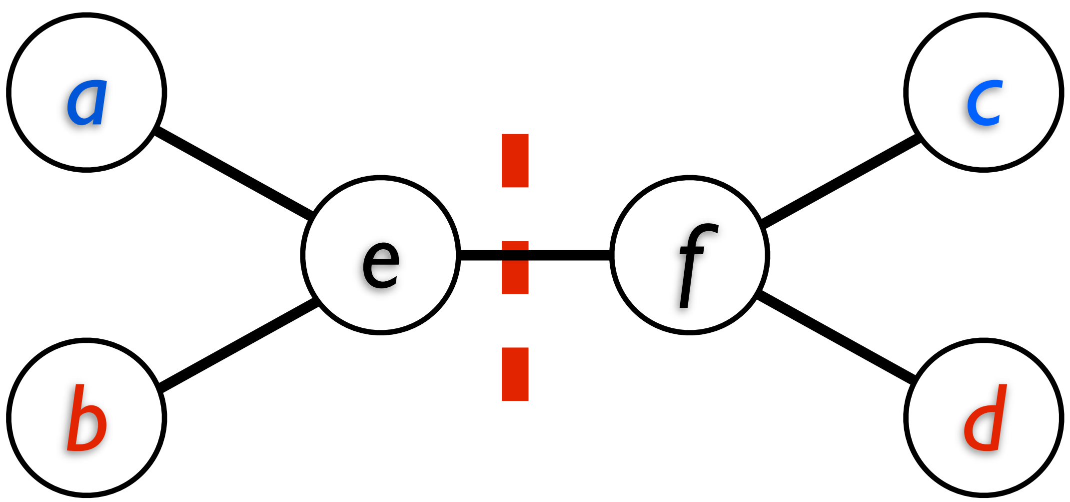


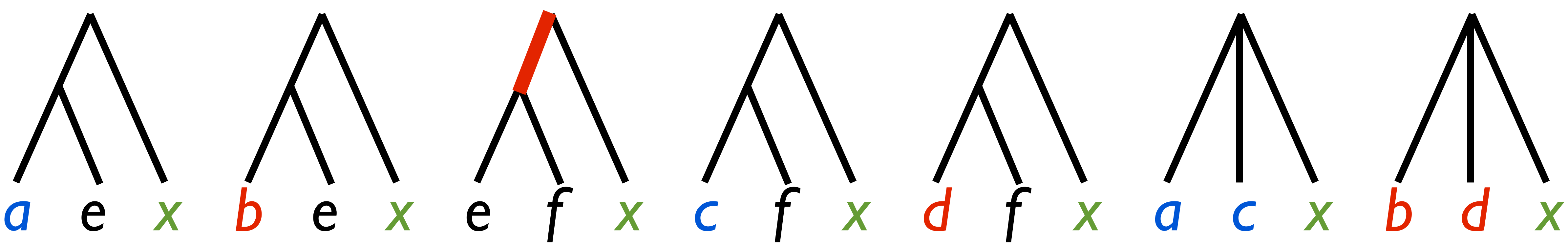
Requests

Reduction from Multicut



Reduction from Multicut





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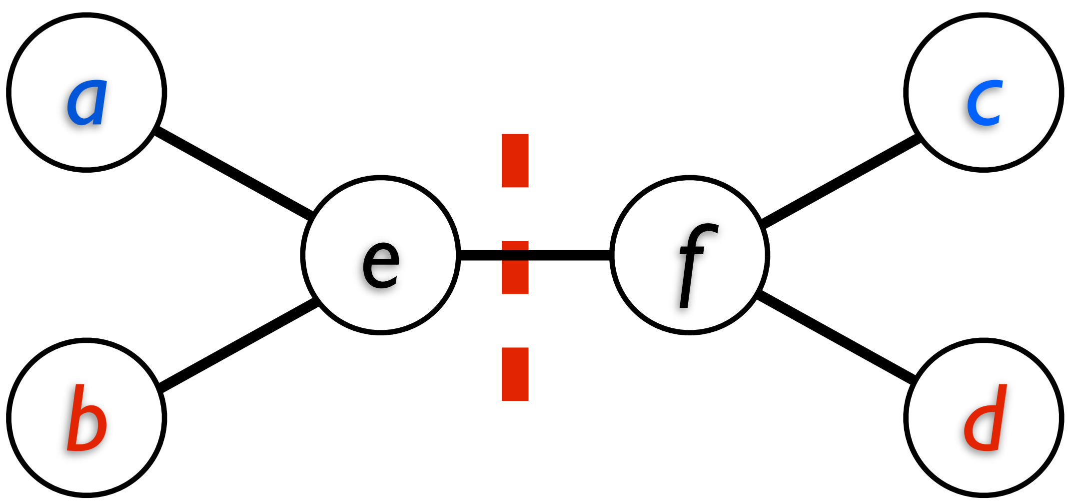

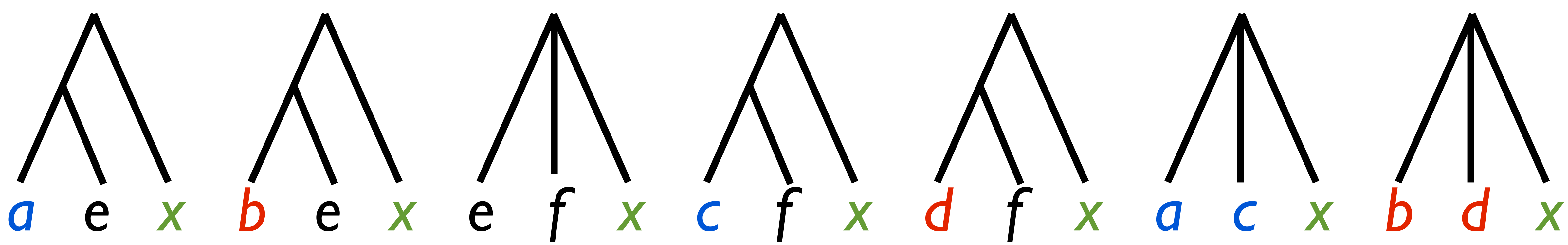

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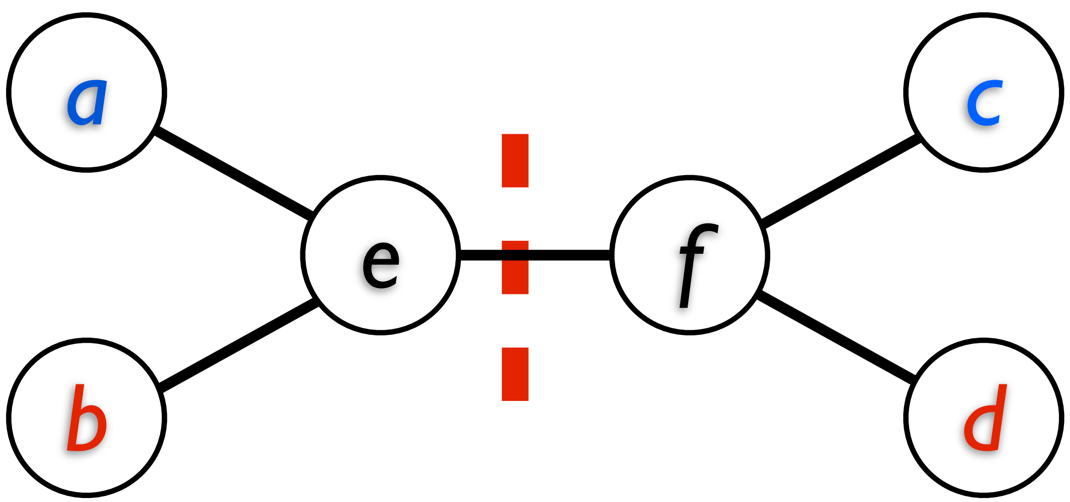

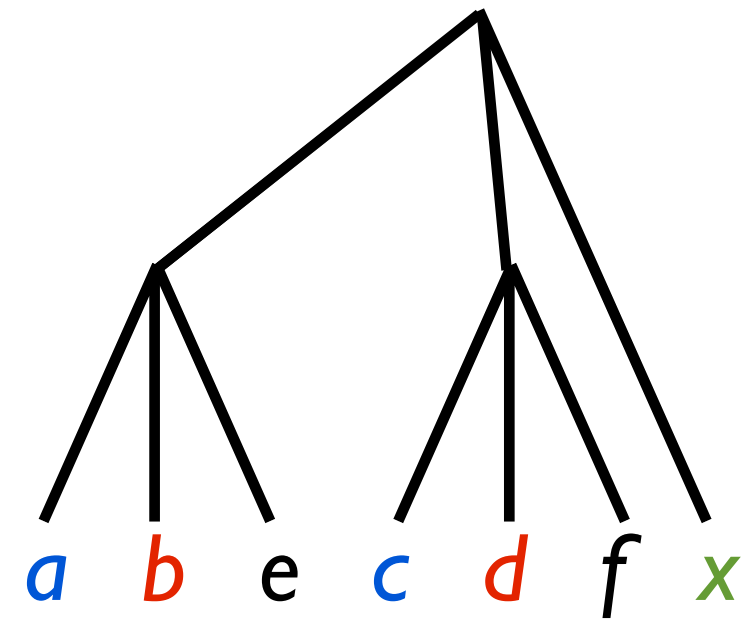

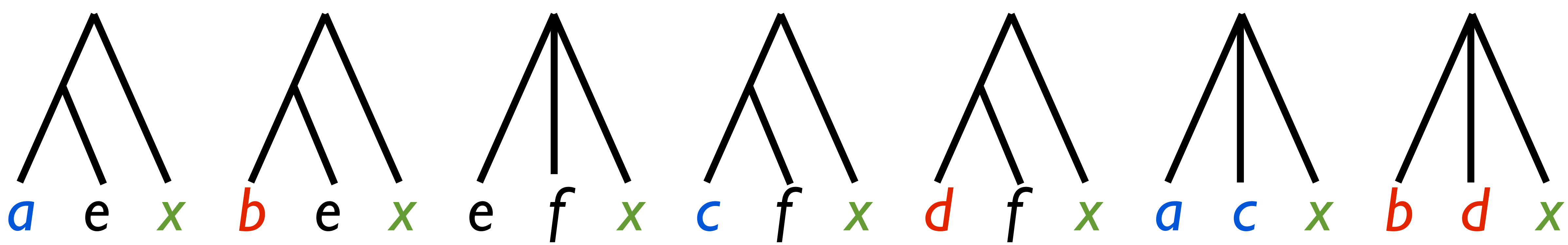

Editing Problems

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- **Approach:** **Edit** P to make it satisfy Π.
  - E.g., contract edges, delete taxa.

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  - X is an ***obstruction*** for Π.

GenericEdit(P,q):

if P satisfies Π **then return** "yes"

if $q == 0$ **then return** "no"

find an obstruction X

if $\exists\ e \in X$ s.t. *GenericEdit*$(P \backslash e, q-1)$ == "yes" **then**
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Fact. If Π can be decided in $n^{O(1)}k^{O(1)}$ time & $|X| \leq ck$, then *GenericEdit* runs in time $(ck)^q \cdot n^{O(1)}k^{O(1)}$.

Strategy

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1. Prove that AST-EC and AST-TR satisfy the small obstruction property:
   If $P = (T_1, T_2, \ldots, T_k)$ has no agreement supertree, there is a "subset" X of P of size $\leq 2k$, such that P restricted to X has no agreement supertree either.

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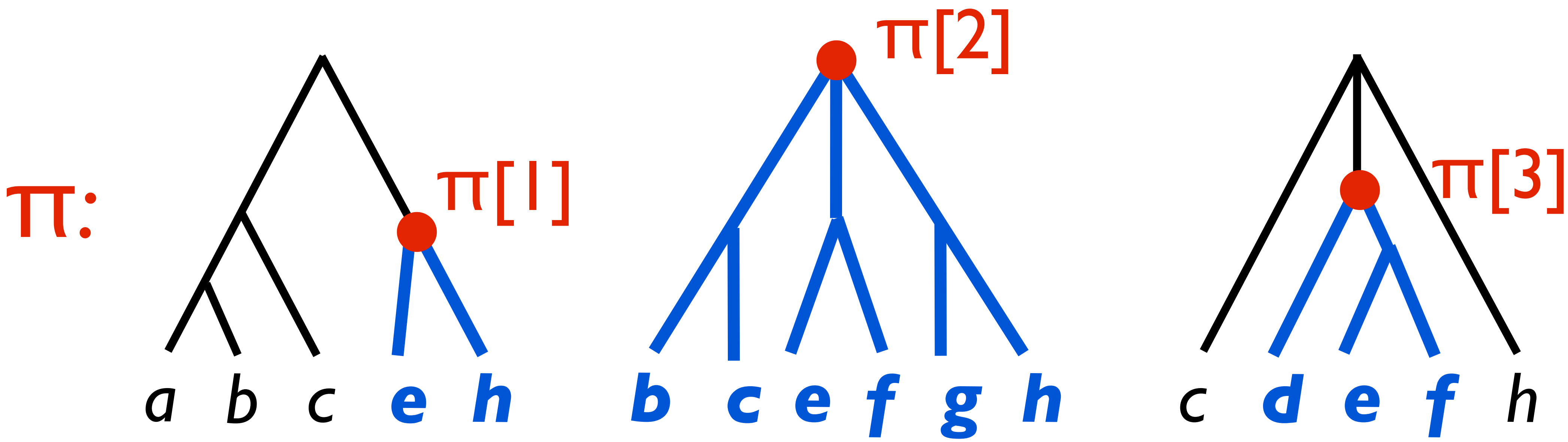
3. Use *GenericEdit* to get $O((2k)^q kn^2)$ algorithm.

Obstructions to Agreement

Position



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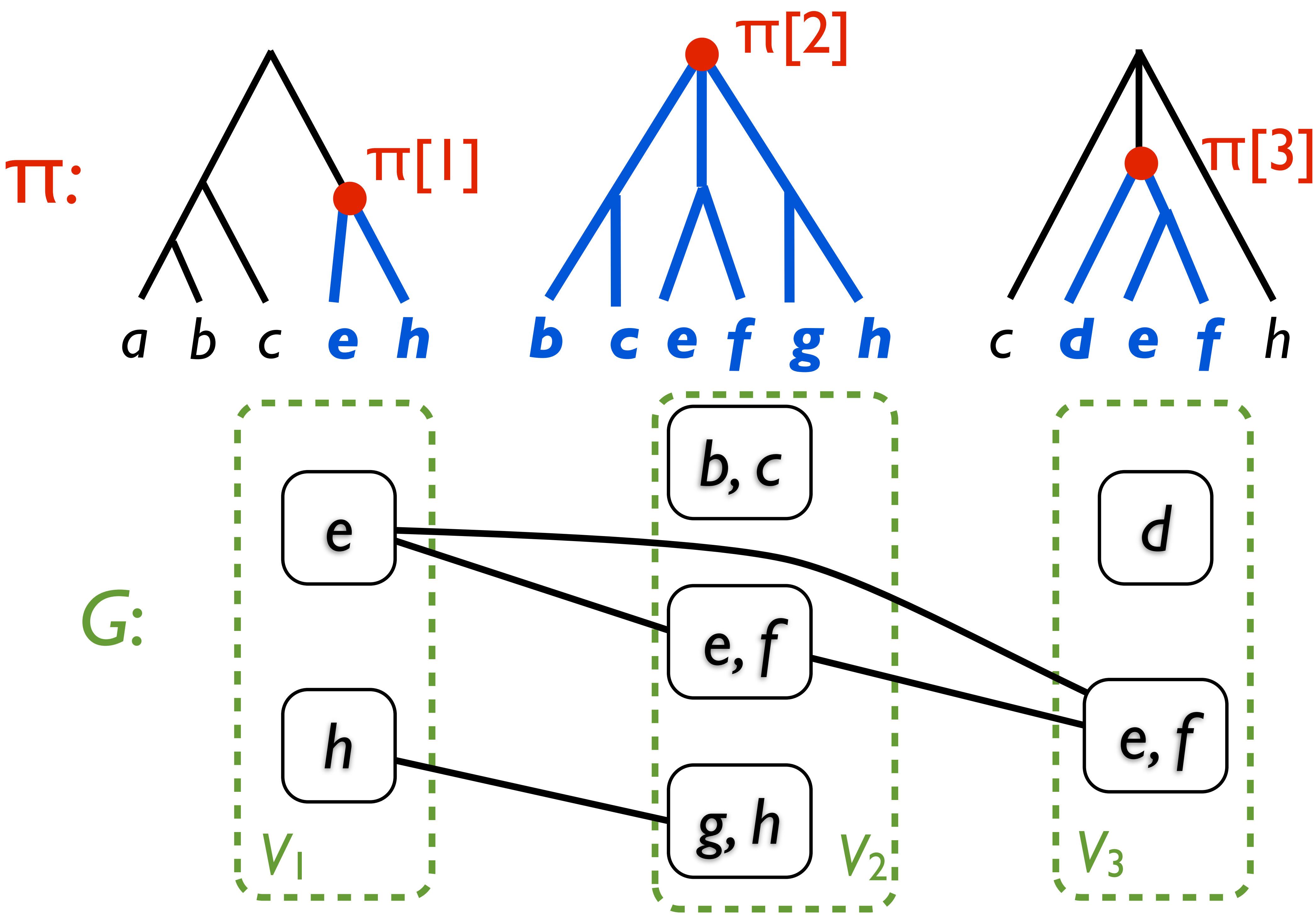

Positions & Agreement

- **Lemma.** *P* has an agreement supertree iff there is an agreement supertree for every position π.
- Algorithm (sketch):
  - Initially π is the root position.
  - If *G* decomposes nicely, find a successor position for each part of the decomposition and recurse.
  - If *G* does not decompose nicely, π encapsulates the taxa on which the trees disagree.

Nice Sets & Successors

- $Y \subseteq V$ is nice if, for each $i \in [k]$, Y contains either zero, one, or all the elements of V_i.
- The successor of $\pi = (\pi[1], \ldots, \pi[k])$ w.r.t. a nice set Y is the vector $\pi_Y = (\pi_Y[1], \ldots, \pi_Y[k])$ where
  - $\pi_Y[i] = \perp$ if $V_i \cap Y = \emptyset$
  - $\pi_Y[i] = p$ if $V_i \cap Y = \{p\}$
  - $\pi_Y[i] = \pi[i]$ if $|V_i \cap Y| \geq 2$

Nice Partitions

- Partition R of V is nice if every set of R is nice, and, for every $\{A, B\} \subseteq R$, A and B are disconnected.
- **Lemma.** The following are equivalent:
  - $\exists$ an agreement supertree for π.
  - $\exists$ a nice partition R of G where $|R| \geq 2$, and, for every $Y \in R$, π_Y has an agreement supertree.
- Partition R is finer than partition Q, denoted $R \leq Q$, iff, for every $C \in R$, $\exists\, D \in Q$ s. t. $C \subseteq D$.
- **Lemma.** The ordering of all nice partitions of G under $\leq$ has a unique minimal element.

Obstructions

- π is an obstructing position if there exists $X \subseteq V$ s.t.
  - $|X \cap V_i| = 2$ for each i, and
  - there is a set $F \subseteq E$ s. t.:
    - $|F| \leq 2|K| - 1$;
    - for each $v \in X$, $\exists\ e \in F$ s. t. $v \in e$; and
    - the minimum nice partition of the subgraph of G induced by F is a singleton.
- X is an obstruction set for π.

Algorithms

- An obstruction set X can be found in $O(kn^2)$ time.
- For AST-EC, can show that at least one edge $\{v, parent(v)\}$ with $v \in X$ must be contracted to achieve agreement.
  - At most $2k$ edges must be considered.
- For AST-TR, can use X to obtain a conflict set of at most $2k - 1$ taxa.
- Solve AST-EC and AST-TR using *GenericEdit*.

Summary

- AST-EC and AST-TR are fixed-parameter tractable in k and p.
  - $O((2k)^p\ kn^2)$-time algorithms.
- Bounds of $2k$ and $2k - 1$ on the respective sizes of the obstruction sets for AST-EC and AST-TR
- Both bounds are tight.

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  - Multicut is fixed-parameter tractable (Marx & Razgon 2011),
  - Is AST-EC fixed-parameter tractable in p only?
- Are AST-EC and/or AST-TR fixed-parameter tractable for unrooted trees?

Funding

- CCF-106029
- DEB-0829674

Thank you!