The Maximum Number of Squares in a Tree

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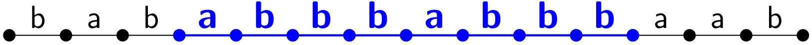
Square

a b b b a b b b

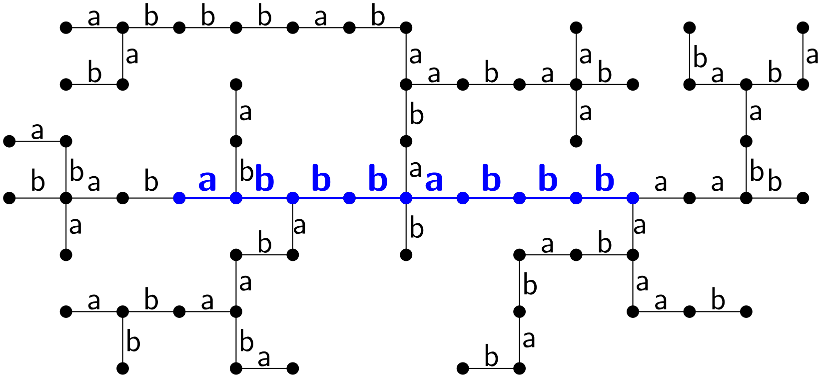
Square in a word

b a b **a b b b a b b b** a a b

Square in a tree



Square in a tree



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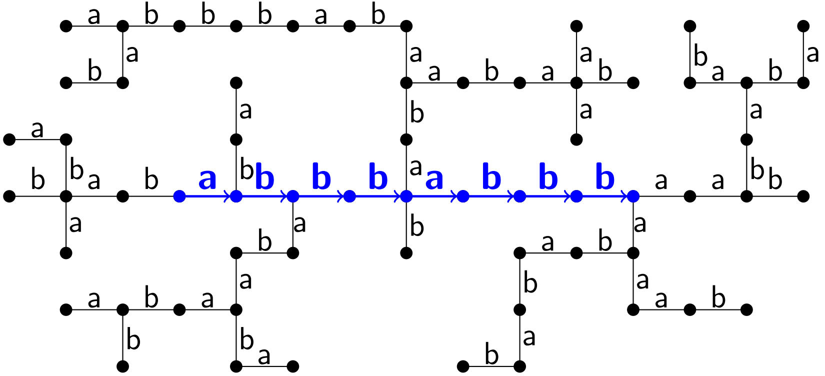

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We consider unrooted, unoriented trees with edges labeled by single letters. Subword of such a tree is a value of a simple path.

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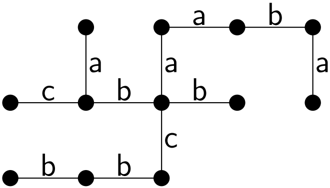

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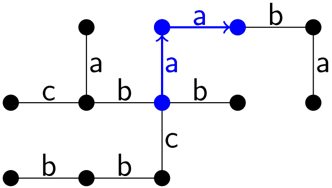

Squares in T: aa

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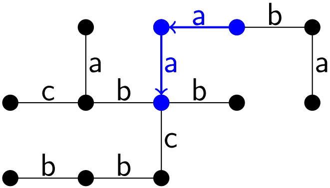

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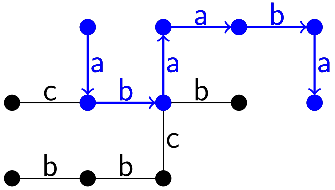

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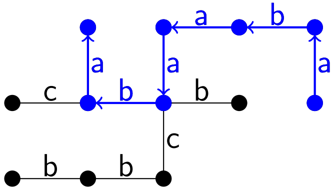

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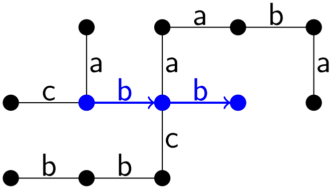

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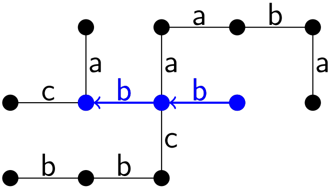

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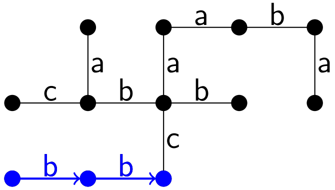

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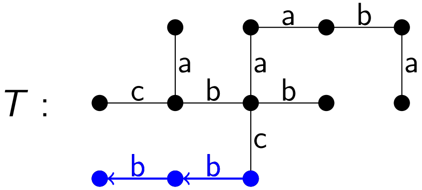

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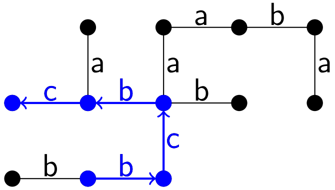

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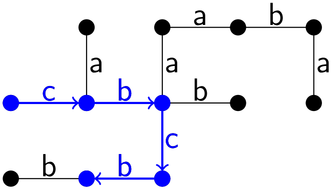

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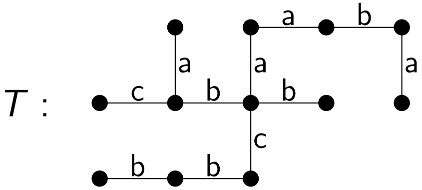

Squares in T: aa, abaaba, bb, bcbc, cbcb.
There are 5 distinct squares, i.e. $\mathrm{sq}(T) = 5$.

Maximum number of squares

What is the maximum number of squares a tree of n nodes might contain?

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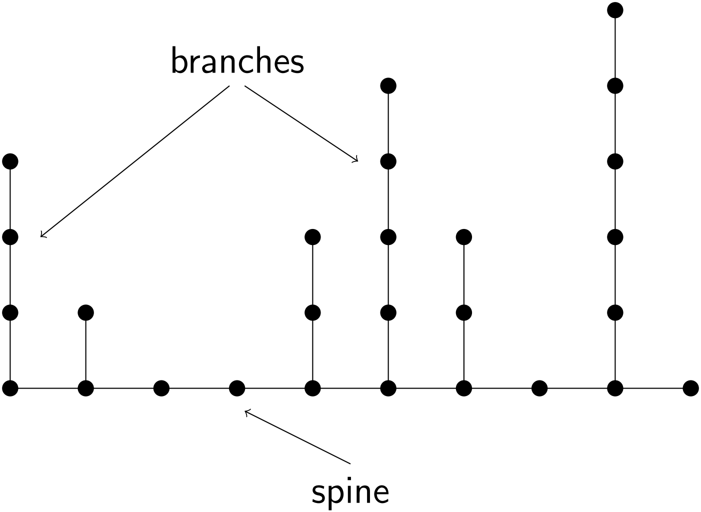
Theorem (Fraenkel & Simpson, 1998)

A word of length n contains at most $2n$ squares. There is a word of length n with $n - o(n)$ squares.

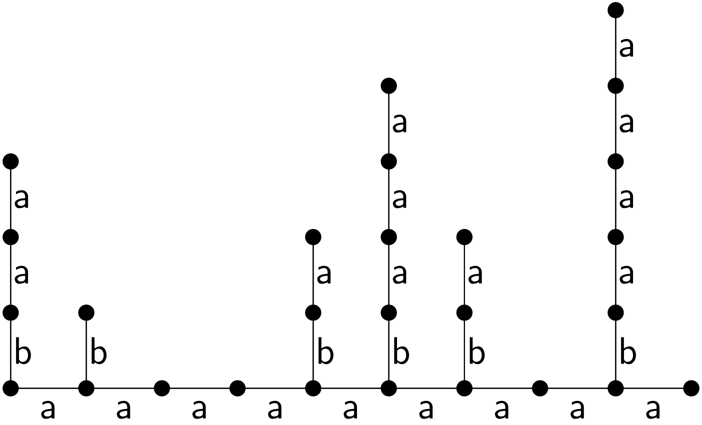
Conjecture

A word of length n contains at most n squares.

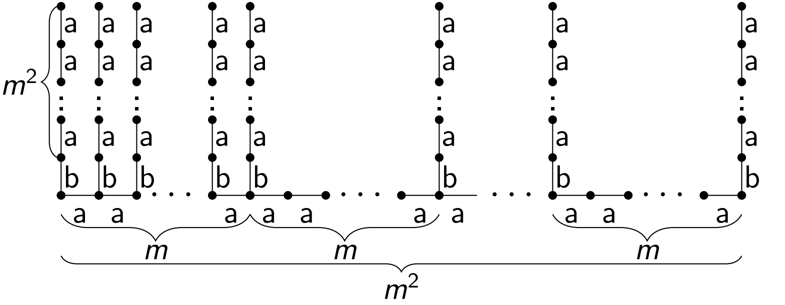
Comb



Standard comb



Lower bound



Branches at $\{0, 1, 2, \ldots, m-1, m, 2m, 3m, \ldots, m^2\}$.
$\Theta(m^3)$ nodes,
$\Theta(m^4)$ squares: $\{a^i b a^{i+j} b a^j : 1 \leq i+j \leq m^2\}$.

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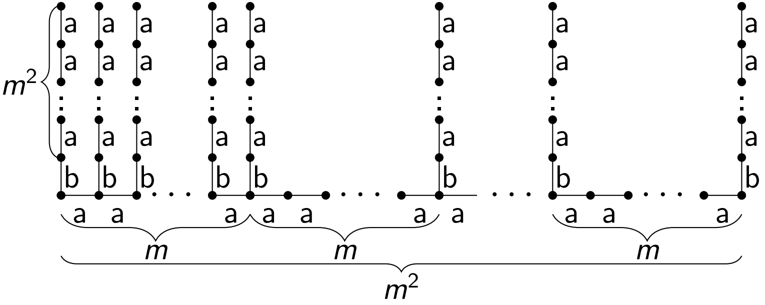

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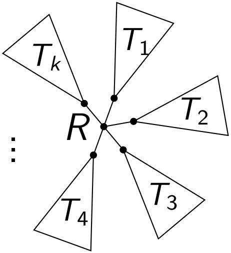
Theorem

There are trees of n nodes with $\Theta(n^{4/3})$ squares.

Centroid decomposition of T



$$|T_i| \leq \frac{|T|}{2}$$

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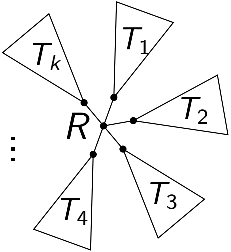

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$\mathrm{sq}(T, R)$ – number of squares passing through R

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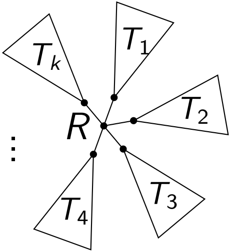

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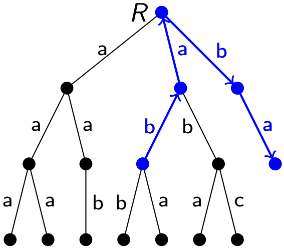
Fact

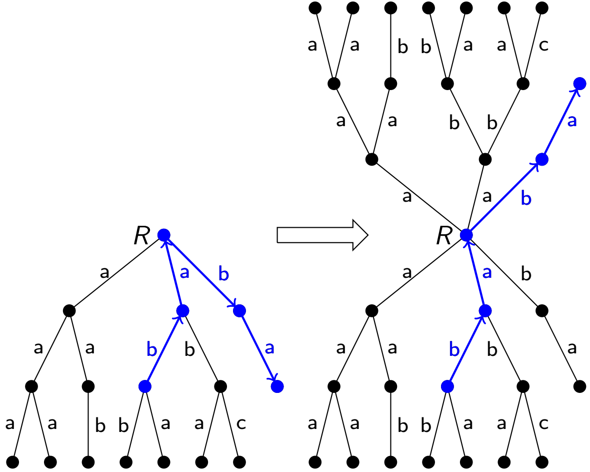
If $sq(T, R) = O(n^{4/3})$ for every tree T of size n, then $sq(T) = O(n^{4/3})$ for every tree T of size n.

D-trees — deterministic double trees



Tree

D-trees — deterministic double trees



Tree Double tree

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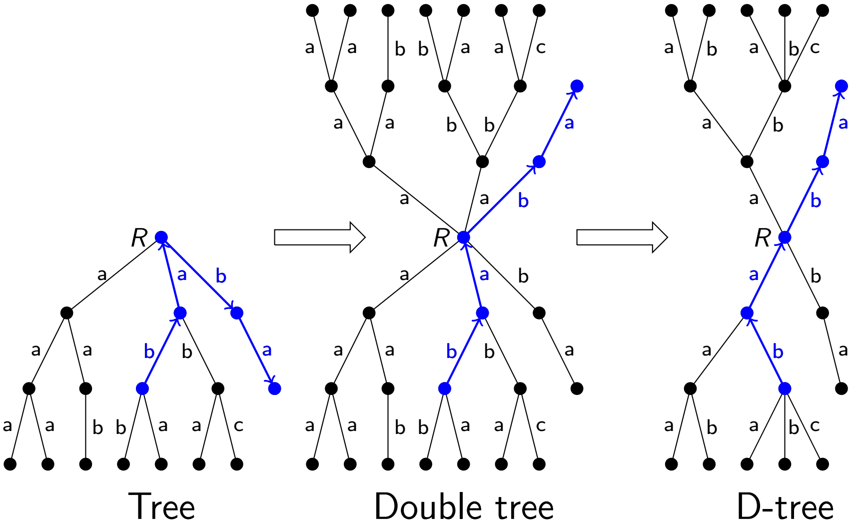
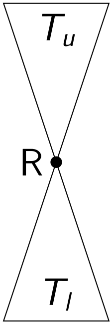


Squares in D-trees



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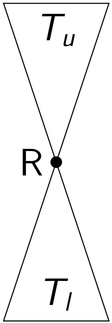
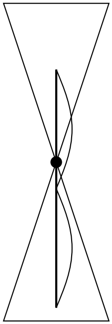


The following lemma implies the main theorem:

Lemma

For any D-tree of size n the number of squares with midpoint in T_l and ending in T_u is $O(n^{4/3})$.

Squares in D-trees



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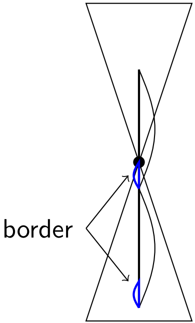

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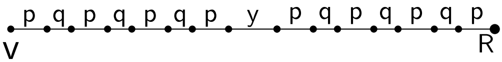
If w has borders of periodic type (p, q) then it has the following representation:

- $w = (pq)^k p$ (global borders) or
- $w = (pq)^l pyp(qp)^r$ (regular borders).

Squares and borders

- Global borders – easy, $O(n)$ in total.
- Regular boders of a single type:



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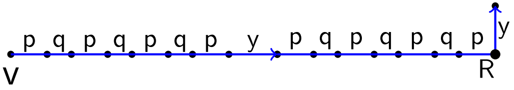

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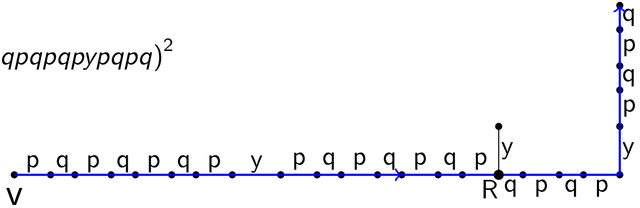

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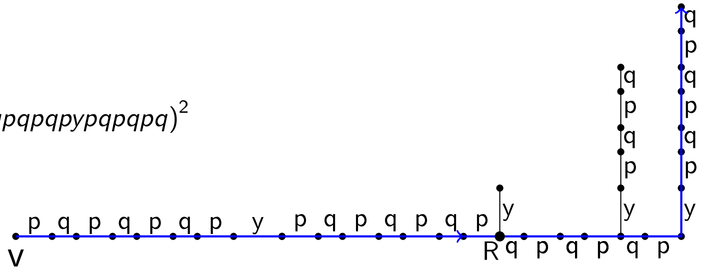
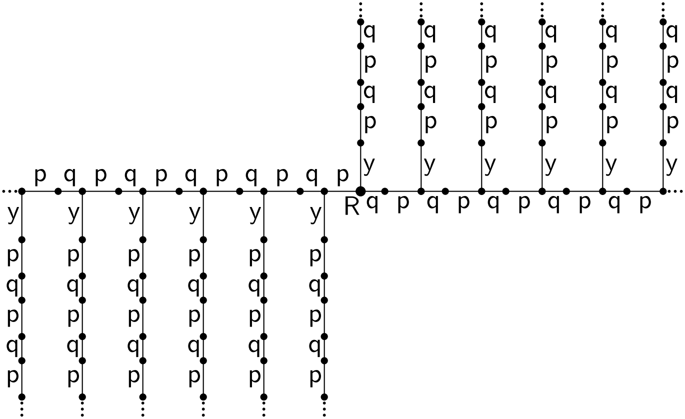
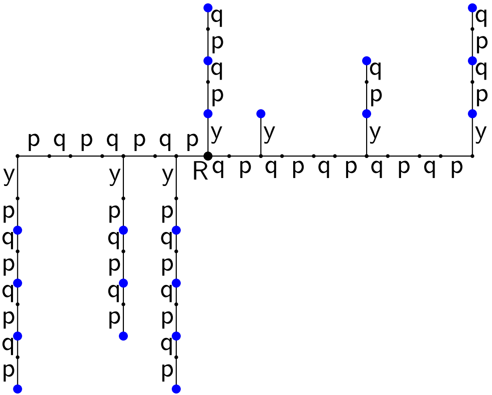


A (p, q, y)-comb of T is the maximal common subtree of T and the following infinite D-tree:



Squares induced by combs



The blue nodes are called *main* nodes of a comb. Squares with both endpoints at these nodes are *induced* by a comb.

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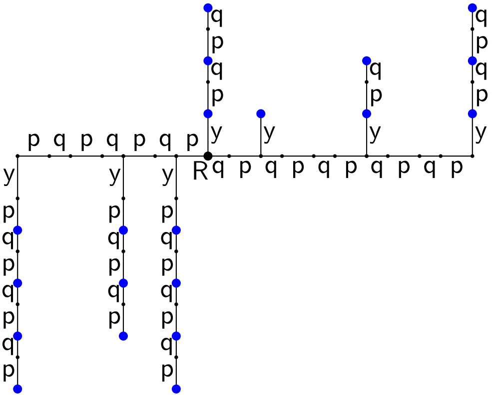

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Size of a comb is the number of main nodes.

Outline of the central proof

1. Just $O(n \log n)$ squares are not induced by combs.
2. Small combs ($\leq n^{0.6}$) induce $o(n^{4/3})$ squares:
   - a comb of size k induces $O(k^{1/2})$ squares starting in a single main node,
   - a single node in T_l can be a main node of $O(\log n)$ combs.
3. Big combs ($> n^{0.6}$) induce $O(n^{4/3})$ squares:
   - combs are almost disjoint in a certain sense: $|Main(\mathcal{C}) \cap Main(\mathcal{C}')| \leq 4$,
   - the total size of big combs is $O(n)$,
   - a comb of size k induces $O(k^{4/3})$ squares.

Thank you

Thank you for your attention!