Approximation Algorithms for Orienting Mixed Graphs

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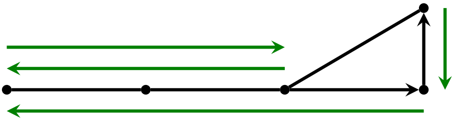
1. What are Maximum Graph Orientations?

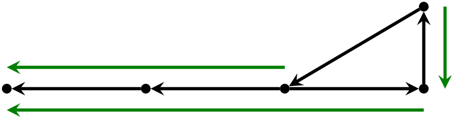
What are Maximum Graph Orientations?

Problem (MAXIMUM-GRAPH-ORIENTATION)

Input *Mixed graph and source-target vertex pairs*



Solution *Oriented graph and satisfied pairs*



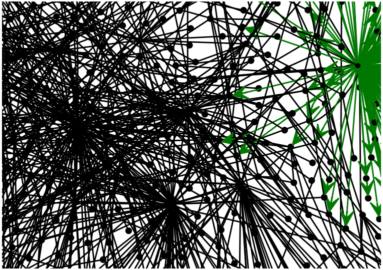
Objective *Maximize number of satisfied pairs*

2 Graph Orientations in Network Biology

Infer Directions of Protein-Protein Interactions

Protein-Protein Interaction Networks

- Interactions are directed in nature
- Directions are not known for all interactions
- Many causal relations are known
- Task: Infer unknown directions from causal relations



Yeast protein-protein interactions and cause-effect pairs

[Silverbush et al., 2011]

3 How to Approximate Maximum Graph Orientations?

Orienting Undirected Graphs

- Previous work focused on undirected graphs
- Corresponds to protein-protein interaction networks without any directionality information

Theorem (Hardness of Approximation [Medvedovsky et al., 2008])

MAXIMUM-GRAPH-ORIENTATION *on undirected graphs is* NP*-hard to approximate within any factor larger than* $11/12$.

Theorem (Approximation Upper Bound [Gamzu et al., 2010])

MAXIMUM-GRAPH-ORIENTATION *on undirected graphs can be approximated in polynomial time within the sub-logarithmic asymptotic factor* $\log\log|vertices| \,/\, \log|vertices|$.

Hardness of Orienting Mixed Graphs

Theorem (Hardness of Approximation)

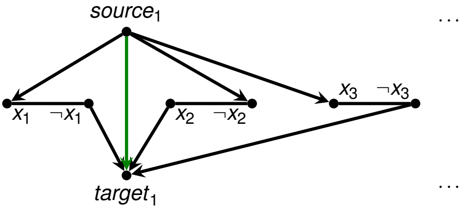
MAXIMUM-GRAPH-ORIENTATION *is* NP*-hard to approximate within any factor larger than* $7/8$*.*

Proof by Reduction.

- Reduce from MAXIMUM-3-SATISFIABILITY

  $(x_1 \lor \neg x_2 \lor x_3) \land \dots$

  to MAXIMUM-GRAPH-ORIENTATION



[Arkin and Hassin, 2002]

- Inapproximability of MAXIMUM-3-SATISFIABILITY [Håstad, 2001]

□

Approximation of Mixed Graph Orientations

Main Theorem (Approximation Upper Bound)

There exists a polynomial-time algorithm that approximates MAXIMUM-GRAPH-ORIENTATION *within the sub-linear asymptotic factor*

$$\frac{1}{(|\mathit{vertices}| + |\mathit{pairs}|)^{0.71} \log |\mathit{vertices}|} \quad .$$

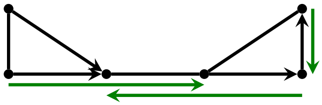
Proof by Algorithm and Analysis

1. Preprocessing that makes the input graph acyclic
2. Greedily satisfy single pairs
3. Satisfy pairs through junction vertex

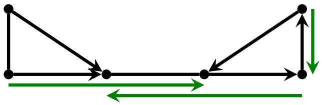
Approximation of Mixed Graph Orientations

1. Preprocessing that makes the input graph acyclic

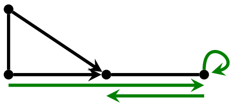
Computation   Mixed graph and pairs



1 Orient cycles



2 Contract strongly connected subgraphs

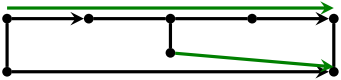


Analysis   Transform solutions back and forth

Approximation of Mixed Graph Orientations

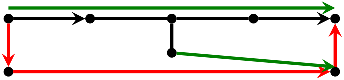
❷ Greedily satisfy single pairs

Computation Acyclic mixed graph and pairs

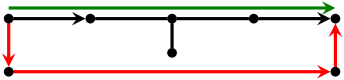


do

Pick pair with shortest path and orient it



Delete disconnected pairs



while (vertices are only crossed by few pairs)

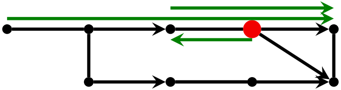
Analysis In every iteration

$$\frac{\textit{win}}{\textit{loss}} \geq \frac{1}{\textit{path length} \times \#\textit{vertex crossing pairs}}$$

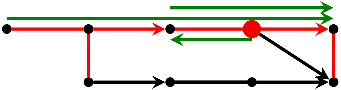
Approximation of Mixed Graph Orientations

❸ Satisfy pairs through junction vertex

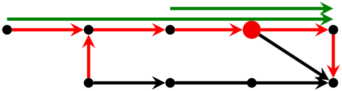
Computation Graph with junction vertex and pairs



1 Compute spanning forward and backward trees



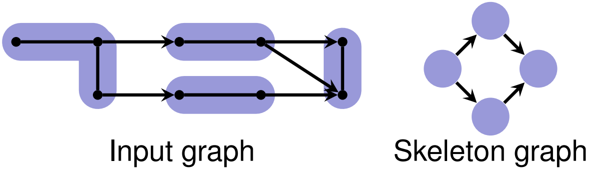
2 Generalize undirected tree orientation from Medvedovsky et al. (2008) to mixed trees



Analysis $\frac{|\textit{crossing pairs}|}{\log|\textit{vertices}|}$ pairs are satisfied

Better Approximations for Tree-Like Instances

Definition (Skeleton graph)



Theorem

If skeleton graphs have . . .

- *. . . feedback vertex number $k \in \mathbb{N}$, then we can satisfy $|pairs|/(4(k+1)\log|vertices|)$ pairs.*
- *. . . tree width $k \in \mathbb{N}$, then we can satisfy $|pairs|/(4(k+1)\log^2|vertices|)$ pairs.*

Proof Idea.

- Decompose into a constant and logarithmic number of junction instances, respectively

Summary and Open Problems

We ...

- ... studied the approximability of orienting mixed graphs; a problem from network biology.
- ... proved a sub-linear approximation ratio.
- ... proved logarithmic and poly-logarithmic approximation ratios for structures instances.

Open Problem

Prove matching upper and lower bounds for orienting undirected and mixed graphs?

References I

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