A Minimal Periods Algorithm with Applications

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Organization

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Repetitions in Words

We assume the alphabet is fixed in the entire discussion.

Definitions

square word of form $w = uu$, $u \neq \epsilon$, e.g., **gaga**.

cube word of form $w = u^3$, $u \neq \epsilon$, e.g., **chachacha**.

(integer) power word of form $w = u^k = \overbrace{uu \cdots u}^{k}$, $u \neq \epsilon$.

fractional power word of form $w = u^k v$, v is a prefix of u, $u \neq \epsilon$. We call $|w|/|u|$ exponent and u period, e.g., **alfalfa**=(**alf**)(**alf**)**a** is a $7/3$ power.

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repetition fractional power with exponent ≥ 2.

square-free word not of the form $w = xuuy$, $u \neq \epsilon$, e.g., **square** is square-free, while **square-free** is not.

X-free for various X is defined similarly.

Research on Repetitions in Words

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- Finding appearance of primitively-rooted squares (Crochemore 1981, Apostolico and Preparata 1983, Main and Lorentz 1984, Franek, Smyth and Tang 2003)
- Finding/counting syntactically distinct maximal repetitions (Slisenko 1983, Gusfield and Stoye 1998, Rytter 2006);
- Finding appearance of maximal repetitions (Main 1989, Kolpakov and Kucherov 1999);

From the Local Point of View

Definitions

The prefix period of w with respect to exponent α and integer s is the shortest word x in $\{ x : |x| > s, w = x^{\beta} y, \beta \geq \alpha \}$. We denote its length by $pp_s^{\alpha}(w)$. If no such x exists, we write $pp_s^{\alpha}(w) = +\infty$.

The strict prefix period with respect to α, s is the shortest word x in $\{ x : |x| > s, w = x^{\alpha} y \}$, and we denote its length by $\overline{pp}_s^{\alpha}(w)$.

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Examples

Let $w = \mathbf{0100101001}$. Then $pp_0^2(w) = 3$, $pp_0^3(w) = +\infty$, $pp_3^2(w) = 5$, $pp_0^{3/2}(w) = 2$, $pp_0^{5/4}(w) = 2$, and $\overline{pp}_0^{5/4}(w) = 8$.

From the Local Point of View (Cont.)

Definitions

The right minimal period array $rmp_s^\alpha(w)$ is defined as

$$rmp_s^\alpha(w)[i] = pp_s^\alpha(w[i\,..\,|w|]),\ \overline{rmp}_s^\alpha(w)[i] = \overline{pp}_s^\alpha(w[i\,..\,|w|]),$$

and the left minimal period array $lmp_s^\alpha(w)$ is defined as

$$lmp_s^\alpha(w)[i] = pp_s^\alpha(w[1\,..\,i]^R),\ \overline{lmp}_s^\alpha(w)[i] = \overline{pp}_s^\alpha(w[1\,..\,i]^R).$$

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Examples

Let $w = \mathbf{0100101001}$. Then

$$rmp_0^2(w) = [3, +\infty, 1, 2, 2, +\infty, +\infty, 1, +\infty, +\infty],$$
$$rmp_1^{3/2}(w) = [2, 3, 5, 2, 2, 2, +\infty, +\infty, +\infty, +\infty],$$
$$\overline{rmp}_1^{3/2}(w) = [2, +\infty, +\infty, 2, 2, 2, +\infty, +\infty, +\infty, +\infty].$$

Finding the Minimal Period Starting at Each Index

Problem

Compute $rmp_s^\alpha(w), lmp_s^\alpha(w), \overline{rmp}_s^\alpha(w), \overline{rmp}_s^\alpha(w)$ *of word* w.

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Global V.S. Local

The computation of $rmp_s^\alpha(w)$ can not be approached directly by the computation of maximal repetition with the same efficiency, since there may be $\Omega(\log |w|)$ primarily-rooted maximal repetitions starting from the same position in w. On the other hand, the maximal repetitions can not be obtained by computing $rmp_s^\alpha(w)$.

Finding the Minimal Period Starting at Each Index

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Research on Repetitions from a Local Point of View

The case rmp_0^2 (Kosaraju 1994); another setting (Duval, Kolpakov, Kucherov, Lecroq, and Lefebvre 2004).

Suffix Tree

Definition

A suffix tree $\mathcal{T}_w$ for $w = w[1..n]$ is a labeled and rooted tree that satisfies the following conditions.

1. All internal nodes excluding the root have ≥ 2 children;
2. labels on edges from same node begin with different letters;
3. there are exactly n leaves, $leaf_i$, and $\tau(leaf_i) = w[i..n]\$$,

where $\tau(v)$ is the concatenation of labels along the path from the root to the node v, and \$ is a special letter not in w.

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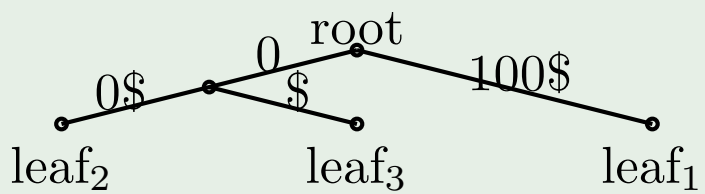
Example (the suffix tree for **100**)



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Construction of $\mathcal{T}_w$ for w can be in linear time (for fixed alphabet) (Weiner 1973, McCreight 1976, Ukkonen 1992).

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Framework of Weiner's Algorithm (Weiner 1973)

begin function `make_suffix_tree(`w`)`
  $T_n \longleftarrow$ suffix tree for $w[n..n]$;
  for i **from** $n-1$ **to** 1 **do** $T_i \longleftarrow$`extend(`T_{i+1}, $w[i..n]$`)` ;
  return T_1 ;
end

Suffix Tree Construction (Cont.)

Framework of Weiner's Algorithm (Cont.)

begin function `extend`(*tree*, $word[i \,..\, n]$)
  find the proper position y in *tree* to insert the new node $leaf_i$;
  if needed, split an edge $x \longrightarrow z$ to $x \longrightarrow y, y \longrightarrow z$ by a new y ;
  create and label the edge $y \longrightarrow leaf_i$ by $word[i + |\tau(y)| \,..\, n]\$$;
end

Details of locating the x, y, z is not relevant and thus omitted.

Suffix Tree Construction (Cont.)

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begin function `extend`(*tree*, *word*$[i \,..\, n]$)
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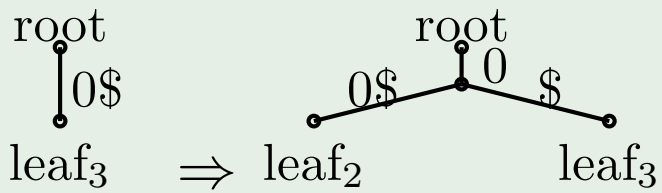

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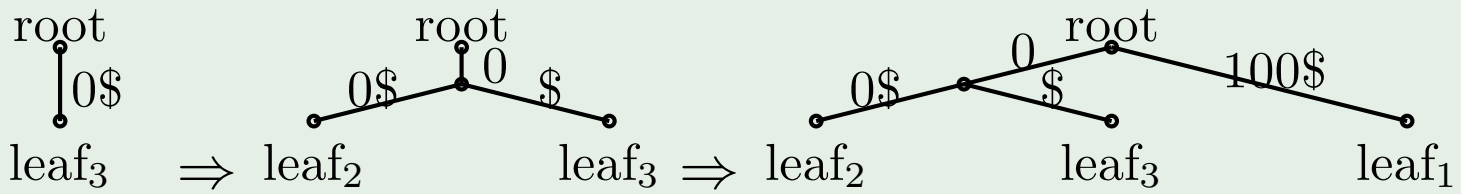
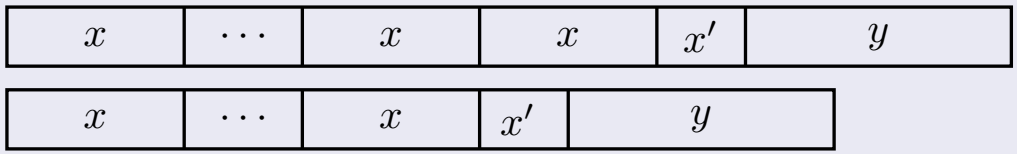


Compute $pp_s^{\alpha}(w)$ from the Suffix Tree of w

Prefix of w has a period x of length $i-1$ with exponent α if and only if $|\mathrm{lcp}(w[1\,..\,|w|], w[i\,..\,|w|])| \geq (\alpha-1)(i-1)$:



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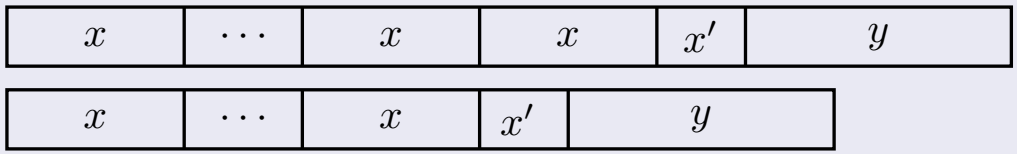


Compute Prefix Period in $O(|w|)$ Time

begin function `compute_pp`(*tree*, *s*, α)
  pp $\longleftarrow +\infty$; preprocessing *tree* for lca operation ;
  foreach $leaf_i$ *other than* $leaf_1$ **do**
    if $|\tau(\mathrm{lca}(leaf_1, leaf_i))| \geq (\alpha-1)(i-1)$ *and* $i-1 > s$ **then**
      if $pp > i-1$ **then** $pp \longleftarrow i-1$;
  return *pp* ;
end

Compute $pp_s^\alpha(w)$ from the Suffix Tree of w (Cont.)

Compute Prefix Period in $O(|w|/\min\{s+1, pp_0^\alpha(w)\})$ Time

begin function `compute_pp`(*tree*, s, α)
  if $\alpha(s+1) > n$ **then return** $+\infty$; **else** $h \longleftarrow leaf_1$;
  while $|\tau(p(h))| \geq (\alpha-1)(s+1)$ **do** $h \longleftarrow p(h)$;
  $pp \longleftarrow +\infty$; preprocessing the tree rooted at h for lca ;
  foreach $leaf_i$ *being a descendent of* h *other than* $leaf_1$ **do**
    if $|\tau(\mathrm{lca}(leaf_1, leaf_i))| \geq (\alpha-1)(i-1)$ *and* $i-1 > s$ **then**
      if $pp > i-1$ **then** $pp \longleftarrow i-1$;
  return pp ;
end

Compute $\overline{pp}_s^\alpha(w)$ from the Suffix Tree of w

Compute Strict Prefix Period in $O(|w|/\min\{s+1, pp_0^\alpha(w)\})$ Time

begin function `compute_pp_strict`(*tree*, *s*, α)
  if $\alpha(s+1) > n$ **then return** $+\infty$; **else** $h \longleftarrow leaf_1$;
  while $|\tau(p(h))| \geq (\alpha-1)(s+1)$ **do** $h \longleftarrow p(h)$;
  $pp \longleftarrow +\infty$; preprocessing the tree rooted at h for lca ;
  foreach $leaf_i$ *being a descendent of* h *other than* $leaf_1$ **do**
    if $|\tau(\mathrm{lca}(leaf_1, leaf_i))| \geq (\alpha-1)(i-1)$ *and* $i-1 > s$ **then**
      if $i-1 \bmod den = 0$ *and* $pp > i-1$ **then** $pp \longleftarrow i-1$;
  return *pp* ;
end

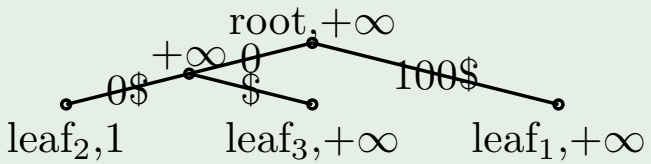
den is the denominator of $\alpha = num/den$, $\gcd(num, den) = 1$.

Suffix Tree with Prefix Periods

Definitions

For given $\alpha > 1, s \geq 0$, a suffix tree with prefix periods is a suffix tree with each node v labeled by $\pi(v) = pp_s^{\alpha}(\tau(v))$. The suffix tree with strict periods is defined analogously by $\overline{\pi}(v) = \overline{pp}_s^{\alpha}(\tau(v)))$.

Example (the suffix tree with (non-strict) prefix periods for **100**)



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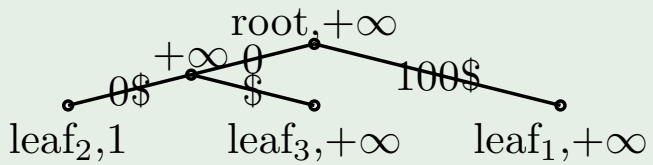

Let u be the parent of v in a suffix tree with prefix periods. 1 If $\pi(u) \neq +\infty$, then $\pi(v) = \pi(u)$. 2 If $\pi(v) \neq +\infty$ and $|\tau(u)| \geq \alpha\pi(v)$, then $\pi(u) = \pi(v)$; otherwise $\pi(u) = +\infty$.

This is also true for suffix tree with strict prefix periods.

Compute $rmp_s^{\alpha}(w)$ Using Suffix Tree

Compute Right Minimal Period Array in $O(|w|^2)$ Time

begin function `compute_rmp`(w, s, α)
  $T_n \longleftarrow$ suffix tree for $w[n..n]$ with $\pi(root), \pi(leaf_n) \longleftarrow +\infty$;
  for i **from** $n-1$ **to** 1 **do**
    $T_i \longleftarrow$`extend`(T_{i+1}, $w[i..n]$) ;
    if *splitting* **then** // y, z are obtained from `extend()`
      if $|\tau(y)| \geq \alpha\pi(z)$ **then** $\pi(y) \longleftarrow \pi(z)$; **else** $\pi(y) \longleftarrow +\infty$;
    if $\pi(y) \neq +\infty$ **then** $\pi(leaf_i) \longleftarrow \pi(y)$;
    else $\pi(leaf_i) \longleftarrow$`compute_pp`($T_i$, s, α) ;
    $rmp[i] \longleftarrow \pi(leaf_i)$;
  $rmp[n] \longleftarrow +\infty$ and **return** rmp ;
end

Compute $rmp_s^{\alpha}(w)$ Using Suffix Tree (Cont.)

Instead of `compute_pp`$(\mathcal{T}_{w[i\,..\,|w|]}, s, \alpha)$, we compute `compute_pp`$(\mathcal{T}_{w[i\,..\,j]}, s', \alpha)$ for some chosen j and s'.

Compute Right Minimal Period Array in $O(|w|^2)$ Time

begin function `compute_rmp`(w, s, α)
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      if $|\tau(y)| \geq \alpha\pi(z)$ **then** $\pi(y) \longleftarrow \pi(z)$; **else** $\pi(y) \longleftarrow +\infty$;
    if $\pi(y) \neq +\infty$ **then** $\pi(leaf_i) \longleftarrow \pi(y)$;
    else $\pi(leaf_i) \longleftarrow$`compute_pp`$(T_i, s, \alpha)$;
    $rmp[i] \longleftarrow \pi(leaf_i)$;
  $rmp[n] \longleftarrow +\infty$ and **return** rmp ;
end

Compute $rmp_s^{\alpha}(w)$ Using Suffix Tree (Cont.)

Compute Right Minimal Period Array, $O(\alpha|w|)$ Time for $s = 0$

begin function `compute_rmp`(w, s, α)

$T_n \longleftarrow$ suffix tree for $w[n\,..\,n]$ with $\pi(root), \pi(leaf_n) \longleftarrow +\infty$;
$A \longleftarrow$ *empty*, $j \longleftarrow n$, and $d \longleftarrow 0$;
for i **from** $n-1$ **to** 1 **do**
  $T_i \longleftarrow$`extend`$(T_{i+1}, w[i\,..\,n])$;
  if *splitting* **then** // y, z are obtained from `extend()`
    if $|\tau(y)| \geq \alpha\pi(z)$ **then** $\pi(y) \longleftarrow \pi(z)$; **else** $\pi(y) \longleftarrow +\infty$;
  if $j - i + 1 > 2\alpha d/(\alpha - 1)$ *or* $|\tau(y)| < d/2$ **then** $A \longleftarrow$ *empty* ;
  if $\pi(y) \neq +\infty$ **then**
    $\pi(leaf_i) \longleftarrow \pi(y)$;
    if $A \neq$ *empty* **then** $A \longleftarrow$`extend`$(A, w[i\,..\,j])$;
  else
    if $A =$ *empty* **then**
      $d \longleftarrow |\tau(y)|$ and $j \longleftarrow i + (\alpha+1)d/(\alpha-1) - 1$;
      $A \longleftarrow$`make_suffix_tree`$(w[i\,..\,j])$;
    else
      $A \longleftarrow$`extend`$(A, w[i\,..\,j])$;
    $\pi(leaf_i) \longleftarrow$`compute_pp`$(A, \max\{s, |\tau(y)|/\alpha\}, \alpha)$;
  $rmp[i] \longleftarrow \pi(leaf_i)$;
$rmp[n] \longleftarrow +\infty$ and **return** rmp ;

end

Compute $rmp_s^\alpha(w)$ Using Suffix Tree — Correctness

In the algorithm, the auxiliary tree A is the suffix tree for $w[i\,..\,j]$.

Theorem

The algorithm is correct.

The only improvement is

$$\pi(leaf_i) \longleftarrow \texttt{compute\_pp}(A,\ \max\{s, |\tau(y)|/\alpha\},\ \alpha)\,.$$

By the construction in the algorithm, $pp_s^\alpha(w[i\,..\,n]) = pp_s^\alpha(w[i\,..\,j])$ and $pp_s^\alpha(w[i\,..\,n]) > |\tau(y)|/\alpha$ holds.

Compute $rmp_s^{\alpha}(w)$ Using Suffix Tree — Complexity

The algorithm is in $O(|w|^2)$ time as the one without using tree A.

Theorem

For $s = 0$, the algorithm is in $O(\alpha|w|)$ time.

When $s = 0$, each call of `compute_pp`$(A, \max\{s, |\tau(y)|/\alpha\}, \alpha)$ is in $O(\alpha)$ time and, by the construction in the algorithm, the total cost in the construction of trees A and tree T is in $O(|w|)$. So the total time is $O(\alpha|w|) + O(|w|) = O(\alpha|w|)$.

Compute $\overline{rmp}_s^{\alpha}(w)$ Using Suffix Tree

Compute Right Minimal Period Array, $O(\alpha|w|)$ Time for $s = 0$

begin function `compute_rmp`(w, s, α)

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for i **from** $n-1$ **to** 1 **do**
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  if *splitting* **then** // y, z are obtained from `extend()`
    if $|\tau(y)| \geq \alpha\pi(z)$ **then** $\pi(y) \longleftarrow \pi(z)$; **else** $\pi(y) \longleftarrow +\infty$;
  if $j - i + 1 > 2\alpha d/(\alpha - 1)$ *or* $|\tau(y)| < d/2$ **then** $A \longleftarrow$ *empty* ;
  if $\pi(y) \neq +\infty$ **then**
    $\pi(leaf_i) \longleftarrow \pi(y)$;
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      $A \longleftarrow$`extend`$(A, w[i\,..\,j])$;
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  $rmp[i] \longleftarrow \pi(leaf_i)$;
$rmp[n] \longleftarrow +\infty$ and **return** rmp ;

end

Pseudo-Powers

In the Setting of DNA Sequence

DNA sequences can be viewed as words over alphabet { **A**, **C**, **G**, **T** }. There is a Watson-Crick complementarity **A**↔**T**, **C**↔**G**, which induces an antimorphic involution θ, e.g., $\theta(\mathbf{AC})$=**GT**.

Definitions

antimorphic involution a function $\theta : \Sigma^* \to \Sigma^*$ that satisfies $\theta(\theta(w)) = w$ and $\theta(uv) = \theta(v)\theta(u)$ for $w, u, v \in \Sigma^*$, e.g., the Watson-Crick complementarity, mirror image (reverse of word).

pseudo-power word of the form $w = u_1 u_2 \cdots u_k$ such that either $u_i = u_j$ or $u_i = \theta(u_j)$ for $1 \leq i, j \leq k$, e.g., **ACGCGTCGTACG**=**ACG**θ(**ACG**)θ(**ACG**)**ACG** is a pseudo 4th power.

Testing Pseudo-Power-Freeness

Algorithms (Chiniforooshan, Kari, Xu 2009)

- Testing pseudo-square-freeness in $O(|w|)$ time;
- Testing pseudo-cube-freeness in $O(|w|^2)$ time;
- Testing pseudo-kth-power-freeness in $O(|w|^2 \log |w|)$ time.

Problem

Testing whether a word w contains any pseudo kth power of the form $\theta(u)\underbrace{uu\cdots u}_{k-1}$ (and of the form $\underbrace{uu\cdots u}_{k-1}\theta(u)$).

Centralized Maximal Pseudo-Palindrome Array

Definition

Let θ be an antimorphic involution. The centralized maximal pseudo-palindrome array cmp_w^θ of word w is defined as $cmp_w^\theta[i] = \max\{m : 0 \leq m \leq \min\{i, |w| - i\}, \theta(w[i - m + 1 .. i]) = w[i + 1 .. i + m]\}$ for $0 \leq i \leq |w|$.

Example

Let $w = \mathbf{0100101001}$ and let θ be the mirror image. Then $cmp_w^\theta = [0, 0, 0, 3, 0, 0, 0, 0, 2, 0, 0]$.

Algorithm (Ilie)

cmp_w^θ for w can be computed in $O(|w|)$ time by constructing the suffix tree for the word $w\$\theta(w)$, where letter $\$$ is not in w.

Testing $\theta(u)u^{k-1}$-Freeness

Algorithm

Testing whether a word w contains factor of the form $\theta(u)\overbrace{uu\cdots u}^{k-1}$ can be done in $O(k|w|)$ time.

Algorithm for Testing $\theta(u)u^{k-1}$-Freeness

1. Computing $rmp_0^{k-1}(w)$;
2. Computing cmp_w^{θ};
3. Testing whether $rmp_0^{k-1}(w)[i] \leq cmp_w^{\theta}[i-1]$ for some $1 \leq i \leq |w|$:

$$\cdots \underline{\theta(u)}\overline{uu\cdots u}\cdots$$

Main Points

We generalize Kosaraju's algorithm for computing minimal squares (i.e. $rmp_0^2(w)$) to computing $rmp_s^\alpha(w)$ and $\overline{rmp}_s^\alpha(w)$ for arbitrary rational exponent $\alpha > 1$ and integer $s \geq 0$. The algorithm is in $O(|w|^2)$ time for arbitrary s and is in $O(k|w|)$ time for $s = 0$.

In particular, the following problem is in linear time: for word w, computing the minimal kth power that begins at each letter in w.

In the end, testing whether a word contains a pseudo kth power of the form $\phi(u)uu\cdots u$ can be done in linear time.

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Try It Yourself

`http://www.csd.uwo.ca/~zhi_xu/demons/cpm2010xu.html`

This is the end. Thank you!