Algorithms for Three Versions of the Shortest Common Superstring Problem

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Classical SCS problem

SCS problem:
Input: words $s_1, s_2, \ldots, s_k \in \Sigma^*$, $|s_1| + \ldots + |s_k| = n$.
Output: the shortest word s containing each s_i as a factor.

Example:
$s_1 = \mathtt{abaab}$, $s_2 = \mathtt{baba}$, $s_3 = \mathtt{aabbb}$, $s_4 = \mathtt{bbab}$.
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A standard assumption: the set $\{s_1, \ldots, s_k\}$ is *factor-free*.

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- 2.59-approximation (Breslauer et al., 1997)
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Multiple occurrences

Notation: $\#occ(u, v)$ – the number of occurrences of the word u as a factor of v.

Input: words $s_1, s_2, \ldots, s_k \in \Sigma^*$, $|s_1| + \ldots + |s_k| = n$.

SUM-SCS(k) problem:
Input: words s_i and positive integer m.
Output: The word shortest u such that $\sum_{i=1}^{k} \#occ(s_i, u) \geq m$.

MULTI-SCS(k) problem:
Input: words s_i and positive integers m_i.
Output: the shortest word u such that $\#occ(s_i, u) \geq m_i$ for $i = 1, 2, \ldots, k$.

MULTI-SCS$_2$(k) problem: for words of length 2.

Assumptions

We assume that $m = O(2^n)$, $m_i = O(2^n)$ and such numbers fit in a single memory cell.
Note that the size of input is $O(n)$.

We wish to find a compressed representation of the result of $O(poly(n))$ size that can be used to reconstruct the actual word in a straightforward manner and in $O(\ell)$ time, where ℓ is the length of the word.

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Sketch of the solution:

1. words $\longrightarrow$ a graph
2. a graph $\longrightarrow$ a deterministic finite automaton
3. a DFA $\longrightarrow$ a regular expression
4. a regexp $\longrightarrow$ a regexp in a normal form
5. a regexp in a normal form $\longrightarrow$ integer program.

Step 1: Prefix graph

$ov(s,t)$ – the largest overlap of words s and t
$pr(s,t)$: $s = pr(s,t)ov(s,t)$

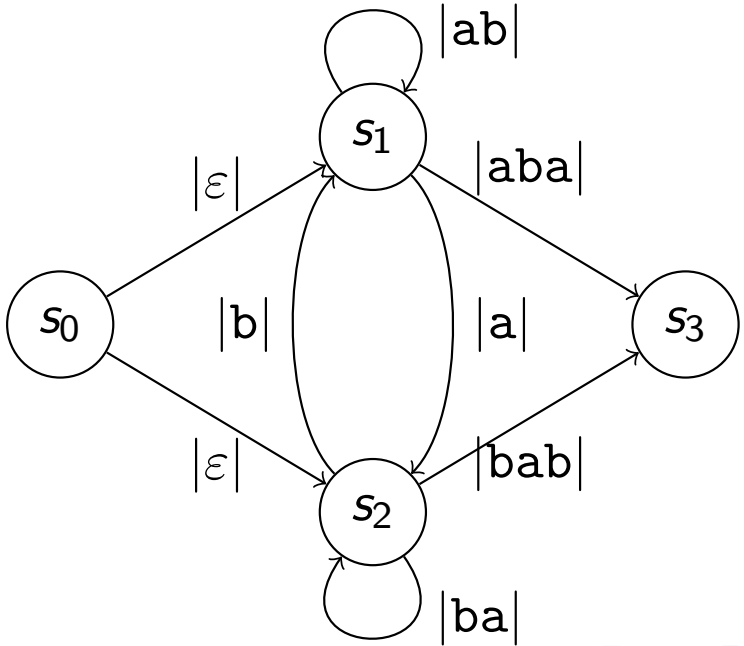
Example: $s = \mathtt{baba}$, $t = \mathtt{abaab}$, $pr(s,t) = \mathtt{b}$, $ov(s,t) = \mathtt{aba}$.

Prefix graph:

Vertices: s_i, the source ($s_0 = \varepsilon$), the sink ($s_{k+1} = \varepsilon$).

Edges: $s_i \xrightarrow{pr(s_i,s_j)} s_j$.

Example for $s_1 = \mathtt{aba}$, $s_2 = \mathtt{bab}$:



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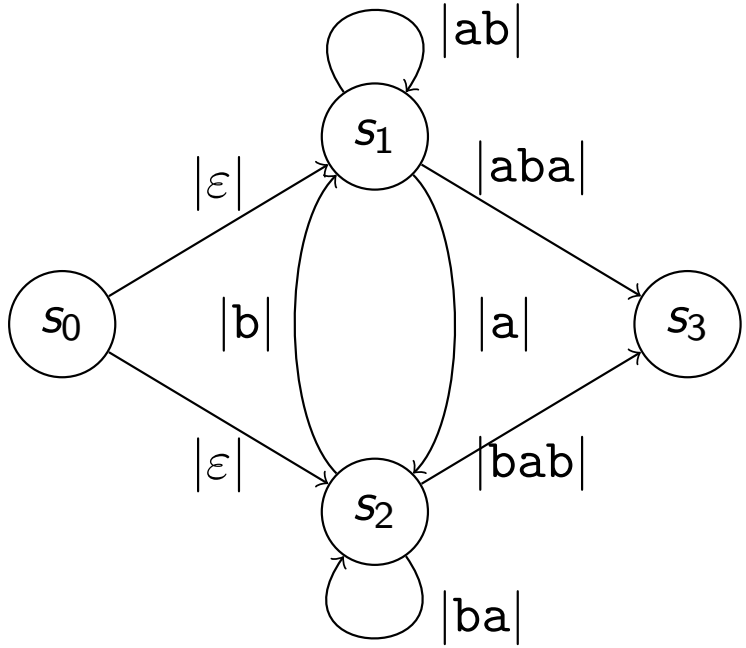

Construction: an optimal $O(n + k^2)$ time algorithm (Gusfield, Landau, Schieber, 1992).

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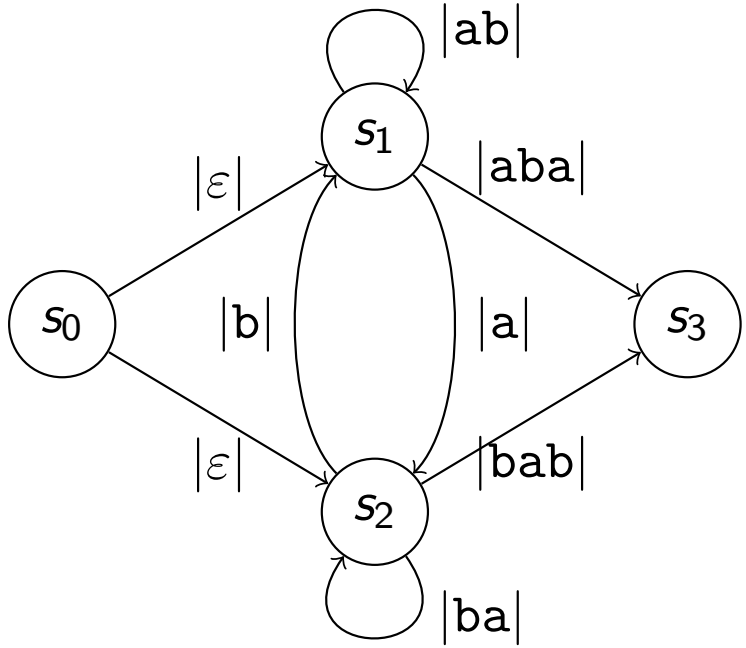

Construction: an optimal $O(n + k^2)$ time algorithm (Gusfield, Landau, Schieber, 1992).

Usage: paths from the source s_0 to the sink s_{k+1} correspond to common superstrings.

Steps 2 and 3: DFA and regexp

Deterministic finite automaton

We treat the prefix graph as a deterministic finite automaton A:

- states: vertices of the graph
- start state: the source (s_0)
- accept state: the sink (s_{k+1})
- alphabet Γ: triples $(i, j, |pr(s_i, s_j)|)$ for edges (s_i, s_j)
- transition function: defined by edges of the graph.

Note that the size of A is $O(poly(k)) = O(1)$.

Regular expression

Let $\alpha(A)$ be a regular expression corresponding to the language accepted by A. Note that its size is $O(1)$ and it can be computed in $O(1)$ time.

Step 4: Normal form

We transform $\alpha(A)$ into a *simpler* regular expression $\beta(A)$ such that:

- $\mathcal{L}(\beta(A)) \subseteq \mathcal{L}(\alpha(A))$
- for every $u \in \alpha(A)$ there exists $v \in \beta(A)$ which is an anagram of u.

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using the transformations (first 1 and 2 repetitively, then 3):

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We obtain the following normal form:

$$\beta(A) = B_1 + B_2 + \ldots + B_k$$

where $B_i = C_{i,1}C_{i,2}\ldots C_{i,l_i}$ and each $C_{i,j}$ is:

- either $a \in \Gamma$
- or $(a_1 a_2 \ldots a_p)^*$, where $a_r \in \Gamma$.

The size of $\beta(A)$ and its computation time is $O(1)$.

Towards step 5

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- Within B_i the only choice is given by a Kleene's star.

Step 5: integer program

Example.

Input words: s_1 and s_2 with $m_1 = 2010$ and $m_2 = 30$

The alphabet: $\Gamma \subseteq \{0, \ldots, 3\} \times \{0, \ldots, 3\} \times (\mathbb{Z}_+ \cup \{0\})$

Consider $B_i = (0,1,0)(1,1,7)^*(1,2,3)\big((2,1,2)(1,1,7)(1,2,3)\big)^*(2,3,5)$

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Let us introduce variables instead of Kleene's stars:

$$(0,1,0)(1,1,7)^{x_1}(1,2,3)\big((2,1,2)(1,1,7)(1,2,3)\big)^{x_2}(2,3,5)$$

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We wish to minimize the expression:

$$len(B_i) = 0 + 7x_1 + 3 + 12x_2 + 5,$$

under the conditions:

$$\begin{aligned} x_1, x_2 &\geq 0 \\ 1 + x_1 + 2x_2 = \#(1, B_i) &\geq m_1 = 2010 \\ 1 + x_2 = \#(2, B_i) &\geq m_2 = 30 \end{aligned}$$

Concluding **MULTI-SCS(k)**

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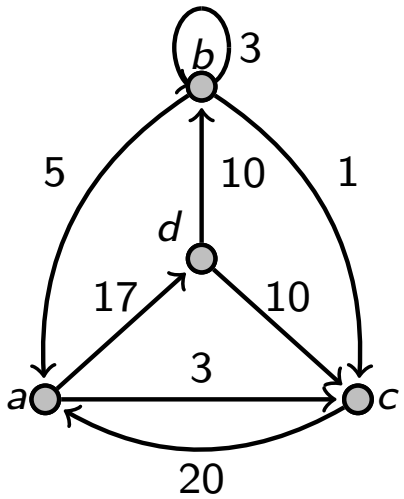
The integer program has $O(1)$ variables and conditions and thus can be solved in polynomial time in the size of input (Lenstra, 1983).

We obtain $O(poly(n))$ time solution for the **MULTI-SCS(k)** problem.

MULTI-SCS$_2(k)$ problem

MULTI-SCS$_2(k)$ problem:
Input: words s_i of length 2 and positive integers m_i.
Output: the shortest word u such that $\#occ(s_i, u) \geq m_i$ for $i = 1, 2, \ldots, k$.

We show an algorithm that computes the length of the word u in $O(n)$ time and computes its compressed representation of size $O(n^2)$ in $O(n^2)$ time (a representation of an Eulerian cycle in a multigraph).



A multigraph for a set of words

$S = \{ad, ac, ba, bb, bc, ca, db, dc\}$

with a sequence of multiplicities

$(m_i)_{i=1}^{8} = (17, 3, 5, 3, 1, 20, 10, 10)$

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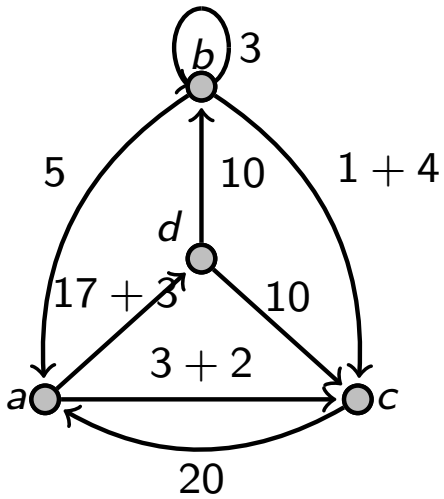

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We present an algorithm with $O(n + k^3 \log m)$ time and $O(n + k^2 \log m)$ space complexity.

We find a shortest path in the prefix graph from s_0 to s_{k+1} containing exactly $m + 1$ vertices. For this we use min-plus product of matrices and repeated squaring starting from the adjacency matrix of the prefix graph.

The compressed representation of the SCS is given by a context-free grammar.

Conclusion

- Contribution to the SCS problem for a fixed number of words
- Assembly of a fixed number of fragments: their number is likely to become huge with Next Generation Sequencing, but their length is limited
- Next step: good approximation for a more general problem