Euler's Method and Midpoint Method - Error

**Euler's Method**

\[ \dot{x}(t) = f(x(t), t) \]

\[ x(t + \Delta t) \approx x(t) + \Delta t f(x, t) \]

**Taylor Series Expansion of \( x(t + \Delta t) \) about \( t \):**

\[ x(t + \Delta t) = x(t) + \frac{\Delta t}{1!} \dot{x}(t) + O(\Delta t^2) \]

- **Constant term**
- **Linear term**
- **Higher order terms**

Euler's Method is equivalent to approximating \( x(t + \Delta t) \) with the constant + linear terms.

Euler's method drops these higher order terms hence making an error of magnitude \( O(\Delta t^2) \) in each time step.

The error in one Euler step is \( O(\Delta t^2) \)

This is called the **local truncation error**

To reach some time \( T \) in the future, we must take \( \frac{T}{\Delta t} \) number of steps.

Therefore the accumulated error is

\[ \frac{T}{\Delta t} \cdot O(\Delta t^2) = O(\Delta t) \]

Thus Euler's Method is said to be **first order accurate**.

Any method which is first order accurate or more accurate is said to be **consistent** with the differential equation.

Euler's Method and the Midpoint Method are both consistent. The Midpoint Method is more accurate.
Midpoint Method

\[ a \quad \Delta x = \Delta t f(x, t) \]
\[ b \quad f_{\text{mid}} = f(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}) \]
\[ c \quad x(t+\Delta t) \approx x(t) + \Delta t f_{\text{mid}} \]

Taylor Series expansion of \( f(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}) \) about \((x, t)\)

\[
f(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}) = f(x,t) + \frac{\Delta t}{2} f_x(x,t) + O(\Delta t^2) + O(\Delta x^2)
\]
\[
+ \frac{\Delta x}{2} \left( f_x(x,t) + \Delta t f_{xx}(x,t) + O(\Delta t^2) \right)
\]
\[
= f(x,t) + \frac{\Delta t}{2} f_x(x,t) + \frac{\Delta x}{2} f_x(x,t) + O(\Delta t^2) + O(\Delta x^2)
\]

Substituting this expression into \( b \), and using \( \Delta t = \Delta x \) from \( a \)

\[
x(t+\Delta t) \approx x(t) + \Delta t f(x,t) + \frac{\Delta t^2}{2} f_x(x,t) + \frac{\Delta t^2}{2} \Delta t f_{xx}(x,t) + O(\Delta t^3)
\]
\[
= x(t) + \Delta t f(x,t) + \frac{\Delta t^2}{2} \left( f_x(x,t) + \frac{\Delta t}{2} f_{xx}(x,t) f(x,t) \right) + O(\Delta t^3)
\]
\[
= x(t) + \Delta t f(x,t) + \frac{\Delta t^2}{2} \Delta t f_{xx}(x,t) + O(\Delta t^3)
\]

Taylor Series expansion of \( x(t+\Delta t) \) about \( t \):

\[
x(t+\Delta t) = x(t) + \Delta t \dot{x}(t) + \frac{\Delta t^2}{2} \ddot{x}(t) + O(\Delta t^3)
\]

Midpoint Method is equivalent to T.S. expansion up to the quadratic term.

These terms differ between Midpoint Method + the T.S. expansion hence Midpoint Method makes an \( O(\Delta t^3) \) error in each time step.

The error in one step of Mid. Method is \( O(\Delta t^3) \).

To reach some time \( T \) in the future, we must take \( \frac{T}{\Delta t} \) steps.

Therefore, the accumulated error is

\[
\frac{T}{\Delta t} \cdot O(\Delta t^3) = O(\Delta t^2)
\]

Thus the Midpoint Method is said to be second order accurate.
Stability of Euler's Method

Euler's Method

\[ x(t + \Delta t) \leq x(t) + \Delta t \, f(x, t) \]

To analyze the stability of a method, we study its behavior for a particular choice of \( f \)

\[ f(x, t) = -kx \quad , \quad k > 0 \]

So our diff. eq. is

\[ \dot{x}(t) = -kx(t) \]

Given the initial value

\[ x(t_0) = x_0 \]

the exact solution to this I.V.P. is

\[ x(t) = x_0 \, e^{-k(t-t_0)} \]

So solution curves should decay as in the figure

Plugging this choice of \( f \) into Euler's Method, we get

\[ x(t + \Delta t) \leq x(t) - \Delta t \, k \, x(t) \]

or

\[ x(t + \Delta t) \leq A(\Delta t) \, x(t), \text{ where} \]

\[ A(\Delta t) = (1 - \Delta t \, k) \]

\( A(\Delta t) \) is called the amplification factor because in each Euler step, we simply multiply (amplify) the previous value of \( x \) by \( A \). Note that \( A \) depends on our choice of time step.
Since the true solution decays, in order to have any hope of faithfully approximating the true solution we should have

$$|A(\Delta t)| < 1$$

so that the numerical solution decays, too.

Therefore, we must choose $\Delta t$ so that

$$|1 - k\Delta t| \leq 1$$

or so that $0 < k\Delta t < 2$

This gives us a time step restriction of

$$\Delta t \leq \frac{2}{k}$$

Note that the larger $k$ is, the smaller time step we are forced to take.