10.1 Curves & Surfaces.

1. Explicit Representation - (limited)
   gives value of dependent variable
   in terms of independent variable.

   E.g.
   \[ y = f(x), \quad \text{or} \]

   \[ x = f(y) \]

2. \[ y = mx + b \]
   Form is not guaranteed to exist for a given curve.
   E.g., vertical line in \( \bullet \).

Example: Circle.

\[ y = (r^2 - x^2)^{1/2} \] half the circle.

\[ y = -(r^2 - x^2)^{1/2} \] other half.

restriction \[ 0 \leq |x| \leq r \]
In 3D, explicit curve representation:

\[ y = f(x) \]
\[ z = g(x) \]

Surface requires 2 indep. variables:

\[ z = f(x, y) \]

Curve or surface may not have explicit representation:

Eg. curve \[ y = ax + b \] can't describe line in constant x-plane.

Eg. surface sphere \[ z = f(x, y) \] given \( x, y \) give \( 0, 1, \) or 2 points on sphere.
Implicit Representations

2D implicit curve

\[ f(x, y) = 0. \]

Line

\[ ax + by + c = 0. \]

Circle

\[ x^2 + y^2 - r^2 = 0. \]

\( f \) is a “testing” or “membership” function but it is not necessarily easy to find a \( y \) satisfying equation, given \( x \).

3D implicit

Surface

\[ f(x, y, z) = 0. \]

e.g. plane

\[ ax + by + cz + d = 0. \]

e.g. sphere

\[ x^2 + y^2 + z^2 - r^2 = 0. \]

curve

intersection of 2 surfaces

\[ f(x, y, z) = 0 \]

\[ g(x, y, z) = 0. \]
Algebraic surfaces:

\[ f(x, y, z) = 0 \]

\( f \) is a sum of polynomials in \( x, y, z \)

E.g. quadric surfaces:

- each term in \( f \) has degree \( \leq 2 \).
- intersections of lines has at most two intersections points.

(useful for factoring rendering).
Parametric Form.

parameter, \( u \)

\[
\begin{align*}
x &= x(u) \\
y &= y(u) \\
z &= z(u)
\end{align*}
\]

curve in 3 dimensions

- similar in 2D

\[
\begin{align*}
x &= x(u) \\
y &= y(u)
\end{align*}
\]

curve in 2 dimensions

Tangent Vector

\[
\frac{dp}{du} = \left( \frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right)
\]

velocity with which the curve is traced out.
Pseudo in the direction tangent to the curve.
Parametric Surfaces

- require 2 parameters, \( u, v \)

\[
\begin{align*}
x &= x(u, v) \\
y &= y(u, v) \\
z &= z(u, v)
\end{align*}
\]

Surface in 3D.

\( p(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \)

\[
\frac{\partial p}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}, \quad \frac{\partial p}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}
\]

tangent vectors define tangent plane

\( \mathbf{N} = \frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v} \)

cross product gives normal.

E.g., Frenet Frame

tangent

normal

binormal.
parametric polynomial Curves

- non-unique.
- widely used in computer graphics

\[ p(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} \]

polynomials

\[ p(u) = \sum_{k=0}^{n} u^k c_k \]

\[ c_k = \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix} \]

\[ p(u) = \sum_{k=0}^{n} u^k \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix} \]

\[ c_k \] independent \( x, y, z \) components.

\( n+1 \) column matrices \( \{ c_k \} \)

coefficients of \( p \).

degrees of freedom in

choosing \( p \).

WLOG, often take \( 0 \leq u, v \leq 1 \)

curve segment

\( 0 \leq u \leq 1 \)

\( \text{Un} \text{Min} \leq u \leq \text{Un} \text{Max} \)

can be rescaled

\[ p(0) \]

\[ p(1) \]
Parametric Polynomial Surfaces

\[ p(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \sum_{i=0}^{n} \sum_{j=0}^{m} C_{ij} u^i v^j \] 3 \( (m+1) (n+1) \) coefficients.

usually take \( m = n \), \( 0 \leq u, v \leq 1 \)

Note: if you hold \( u \) constant, get a parametric curve in \( v \), \( v \) vice versa.

\( \begin{array}{c}
\text{Surface patch}
\end{array} \)

\( v = 1 \)

\( u = 0 \)

\( u = \infty \)


Design Considerations

- local control of shape
- smoothness and continuity
- ability to evaluate derivatives
- stability
- ease of rendering.

E.g. model airplane

Cross section

approximation out of number of wood strips.
join points.
For polynomial curves and surfaces, smooth segments are smooth and have all derivatives. Complications may arise at join points.

- We want local control, so that we want to design each segment individually, interactively.
- We want stability: small changes in values of input parameters should cause only small change in output.

Typically, consider data at a small set of control points.

Curve may interpolate control points or not directly interpolate, but come close to all.
Parametric Cubic Polynomial Curves

- high degree + many parameters
- evaluating pts. on curve costly
- higher order - more oscillatory

Use low degree polynomials, defined over short interval.

Piecewise cubic polynomials

\[ p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \sum_{k=0}^{3} c_k u^k \]

\[ = \begin{pmatrix} c_{0x} \\ c_{0y} \end{pmatrix} + \begin{pmatrix} c_{1x} \\ c_{1y} \end{pmatrix} u + \begin{pmatrix} c_{2x} \\ c_{2y} \end{pmatrix} u^2 + \begin{pmatrix} c_{3x} \\ c_{3y} \end{pmatrix} u^3 \]

\[ = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix} = C u \]

For each \( x_i, y_i \), we have 4 unknowns \( c_0, c_1, c_2, c_3 \)

\[ \rightarrow \text{need 4 equations} \]

- might have interpolating conditions
  - continuity conditions at joint points

- or conditions that curve pass close to data points.

Each type of condition defines different curve from same data.
10.4 Interpolation

Cubic interpolating polynomial
- not necessarily commonly used, but illustrates principles.

4 control points: \( P_0, P_1, P_2, P_3 \)

\[
P_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}
\]

Seek \( C \), st. \( C \) passes through \( P_0, \ldots, P_3 \)

i.e. interpolate.

\( P_0, \ldots, P_3 \) give 12 equations

Assume \( P_0 = p(0) \), \( P_1 = p(\frac{1}{3}) \), \( P_2 = p(\frac{2}{3}) \), \( P_3 = p(1) \)

equally spaced.

Conditions:

\[
P_0 = p(0) = c_0
\]

\[
P_1 = p(\frac{1}{3}) = c_0 + \frac{1}{3} c_1 + \left(\frac{1}{3}\right)^2 c_2 + \left(\frac{1}{3}\right)^3 c_3
\]

\[
P_2 = p(\frac{2}{3}) = c_0 + \frac{2}{3} c_1 + \left(\frac{2}{3}\right)^2 c_2 + \left(\frac{2}{3}\right)^3 c_3
\]

\[
P_3 = p(1) = c_0 + c_1 + c_2 + c_3
\]

For each \( x, y, z \),

\[
p = A c
\]

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
c = \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{bmatrix}
\]
Or, together

\[
\begin{pmatrix}
p_{1x} & p_{1y} & p_{1z} \\
p_{2x} & p_{2y} & p_{2z} \\
p_{3x} & p_{3y} & p_{3z}
\end{pmatrix}
= \begin{pmatrix}
A_{1x} & A_{1y} & A_{1z} \\
A_{2x} & A_{2y} & A_{2z} \\
A_{3x} & A_{3y} & A_{3z}
\end{pmatrix}
\begin{pmatrix}
p_{0x} & p_{0y} & p_{0z} \\
p_{1x} & p_{1y} & p_{1z} \\
p_{2x} & p_{2y} & p_{2z}
\end{pmatrix}
\]

This is a non-singular system, and we can invert it

\[ M_I = A^{-1} \]

\[ C = M_I P \]

2 points unique degree 1 poly. curve

3 points unique degree 2.

Rather than degree n polynomial, set of cubic polynomials
E.g.

- First use $p_0, \ldots, p_3$ to get one cubic.
- Then use $p_3, \ldots, p_6$ to get another cubic.

⇒ Continuity at the join point.

Note: $A$ and hence $M_2$ is the same for each segment if we take $u = 0, \frac{1}{3}, \frac{2}{3}, 1$ for each segment.

Problem: derivatives at the join points discontinuous.

- May get something like this.
10.4.1 Blending Functions

\[ p(u) = u^T C \]

\[ p(u) = (x(u), y(u), z(u)) = \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \]

Recall we determined \( C \) as

\[ C = M_I P \]

so substituting

\[ p(u) = u^T (M_I P) \]

\[ p(u) = (u^T M_I) P \]

depends on data

data-independent

\[ p(u) = b(u)^T P \]

where

\[ b(u) = M_I^T u \]

blending polynomials

\[ b(u) = \begin{pmatrix} b_0(u) \\ b_1(u) \\ b_2(u) \\ b_3(u) \end{pmatrix} \]

blend together effect of different control points
Blending functions

Lagrange basis polynomials

\[ \begin{align*}
    b_0(u) &= -\frac{9}{2} (u - \frac{1}{3})(u - \frac{2}{3})(u-1) \\
    b_1(u) &= \frac{27}{2} u (u - \frac{2}{3})(u-1) \\
    b_2(u) &= -\frac{27}{2} u (u - \frac{1}{3})(u-1) \\
    b_3(u) &= \frac{9}{2} u (u - \frac{1}{3})(u - \frac{2}{3})
\end{align*} \]

Note:
\[
\begin{align*}
    b_0(0) &= -\frac{9}{2} (-\frac{1}{3})(-\frac{2}{3})(+1) = \frac{18}{18} = 1 \quad \checkmark \\
    b_1\left(\frac{1}{3}\right) &= \frac{27}{2} \left(\frac{1}{3}\right) \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) = \frac{27 \cdot 2}{2 \cdot 2 \cdot 2} = 1 \quad \checkmark \\
    b_2\left(\frac{2}{3}\right) &= +\frac{27}{2} \frac{2}{3} \frac{1}{3} \frac{1}{3} = \frac{27 \cdot 2}{2 \cdot 2 \cdot 2} = 1 \quad \checkmark \\
    b_3(1) &= \frac{9}{2} 1 \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) = \frac{9 \cdot 2}{2 \cdot 3 \cdot 3} = 1 \quad \checkmark
\end{align*}
\]

Can be derived this way:

- blend polynomials for interpolation

Problems:
- already quite oscillatory looking — worse for higher order
- kinks at join points.
10.4.2 Cubic Interpolating Patch.

natural extension of interpolating curve

Bicubic Surface Patch

\[ p(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} u^i v^j c_{ij} \]

\[ p(u,v) = u^T C v, \quad u = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ v \\ v^2 \\ v^3 \end{pmatrix} \]

\( C \) has \( 4 \times 4 = 16 \) elements for each dimension \((x,y,z)\) for a total of \( 48 \) elements. 

16 control points.

\[ V = 0 \]

\[ P_{00} \quad P_{01} \quad P_{02} \quad P_{03} \]

\[ P_{10} \quad P_{11} \quad P_{12} \quad P_{13} \]

\[ P_{20} \quad P_{21} \quad P_{22} \quad P_{23} \]

\[ P_{30} \quad P_{31} \quad P_{32} \quad P_{33} \]

To solve, first consider \( V = 0 \).

\[ p(u,0) = \sum_{i=0}^{3} u^i 0^j c_{ij} = \sum_{i=0}^{3} u^i c_{i0} \]

Curve in \( u \) that interpolates \( P_{00}, P_{10}, P_{20}, P_{30} \).
\[ p(u, v) = (u^T M_\mathbf{I}) P_0 \]
\[ = u^T M_\mathbf{I} \begin{bmatrix} P_{00} \\ P_{10} \\ P_{20} \\ P_{30} \end{bmatrix} \]
\[ = u^T C \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

Similarly for \( p(u, \frac{1}{3}) \), \( p(u, \frac{2}{3}) \), \( p(u, 1) \)

Write all 16 equations as:

\[ u^T M_\mathbf{I} P = u^T C A^T \]
\[ A = M_\mathbf{I}^{-1} \]

\[ \Rightarrow C = M_\mathbf{I} P M_\mathbf{I}^T \]

\[ p(u, v) = u^T (M_\mathbf{I} P M_\mathbf{I}^T) v \]
\[ = u^T M_\mathbf{I} P M_\mathbf{I}^T v \]
\[ = (M_\mathbf{I}^T u)^T P (M_\mathbf{I}^T v) \]
\[ = b(u)^T P b(v) \]
\[ = \sum_{i=0}^{3} \sum_{j=0}^{3} b_i(u) b_j(v) \]

\[ b_i(u) b_j(v) \] Blending patch.

control points are weighted by blending patches.
Algebraic form:

\[ p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \mathbf{u}^\top \mathbf{c} \]

Geometric form:

\[ p(u) = F(u) p_0 + F_1(u) p_1 + F_2(u) p_2 + F_3(u) p_3 \]

\[ p(u) = \mathbf{b}(u)^\top \mathbf{p} \]

Coefficients from data

\[ \mathbf{c} = \mathbf{M}_I \mathbf{p} \]

Interpolation

\[ p(0) = c_0 \]
\[ p(1) = c_0 + c_1 + c_2 + c_3 \]
\[ p\left( \frac{1}{3} \right) = c_0 + \left( \frac{1}{3} \right) c_1 + \left( \frac{1}{3} \right)^2 c_2 + \left( \frac{1}{3} \right)^3 c_3 \]
\[ p\left( \frac{2}{3} \right) = c_0 + \left( \frac{2}{3} \right) c_1 + \left( \frac{2}{3} \right)^2 c_2 + \left( \frac{2}{3} \right)^3 c_3 \]

\[ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \left( \frac{1}{3} \right)^2 & \left( \frac{1}{3} \right)^3 \\ \frac{2}{3} & \frac{1}{3} & \left( \frac{2}{3} \right)^2 & \left( \frac{2}{3} \right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \]

\[ p = \mathbf{A} \mathbf{c} \]
\[ \mathbf{c} = \mathbf{M}_I ^{-1} \mathbf{p} \]
Algebraic \[ p(u) = u^T c \] 

Geometric \[ p(u) = b(u)^T p \] \[ p \text{ in terms of } \begin{array}{c} \text{coefficients} \\ \text{data} \end{array} \]

\[ M_p = \frac{1}{c} \]

\[ p(u) = u^T M_p = (u^T M)_p \]

\[ b(u) = M^T u \]
\[ p(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} \] Cubic Interpolating Curves + Surfaces
(treating \( x, y, z \) separately).

\[ x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \]

\[ x(u) = u^T c \]

Interpolation of points \( P_0, P_1, P_2, P_3 \)
\((0, x_0), (\frac{1}{3}, x_1), (\frac{2}{3}, x_2), (1, x_3)\)

\[
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & (1/3)^2 & (1/3)^3 \\ 2/3 & (2/3)^2 & (2/3)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

4 equations

4 unknowns.

\( p \)

\( A \)

\( c \)

\( A \) is a Vandermonde matrix.

Let \( M_z = A^{-1} \)

Then \( c = M_z p \) coefficients.

This approach corresponds to using a monomial basis.
It is also possible to use a Lagrange basis

\[ \chi(u) = b_0(u)X_0 + b_1(u)X_1 + b_2(u)X_2 + b_3(u)X_3 \]

where

\[ b_0(u) = \frac{(u-\frac{1}{3})(u-\frac{2}{3})(u-1)}{(0-\frac{1}{3})(0-\frac{2}{3})(0-1)} = \frac{-9}{2} (u-\frac{1}{3})(u-\frac{2}{3})(u-1) \]

\[ b_1(u) = \frac{(u-0)(u-\frac{2}{3})(u-1)}{(\frac{1}{3}-0)(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-1)} = \frac{27}{2} (u)(u-\frac{2}{3})(u-1) \]

\[ b_2(u) = \frac{(u-0)(u-1)(u-1)}{(\frac{2}{3}-0)(\frac{2}{3}-\frac{1}{3})(\frac{2}{3}-1)} = \frac{-27}{2} (u-\frac{1}{3})(u-1) \]

\[ b_3(u) = \frac{(u-0)(u-\frac{1}{3})(u-\frac{2}{3})}{(1-0)(1-\frac{1}{3})(1-\frac{2}{3})} = \frac{9}{2} (u-\frac{1}{3})(u-\frac{2}{3}) \]
\[ x(u) = u^T \mathcal{C} = u^T M \mathcal{X} \]
= \( \beta(u)^T x \) \hspace{1cm} \text{blending functions (or Lagrange polynomials)}

\[ x(u) = b_0(u)x_0 + b_1(u)x_1 + b_2(u)x_2 + b_3(u)x_3 \]

\[ x(u, v) = \sum_{i=0}^{3} \sum_{j=0}^{3} u^i v^j C_{ij} = u^T C y \]

Assume our data is given at \( u = 0, \frac{1}{3}, \frac{5}{3}, 1 \)
\( v = 0, \frac{1}{3}, \frac{5}{3}, 1 \)

16 qgs., 16 unknowns
16 \times 16 linear system.

\[ \Rightarrow \text{But each curve is also an interpolating cubic} \]

\[ \Rightarrow \text{decouples into 4, 4 \times 4 systems.} \]

\[ \begin{align*}
x(u, 0) &= U^T M_y \begin{pmatrix} x_{00} \\ x_{10} \\ x_{20} \\ x_{30} \\ x_{01} \\ x_{11} \\ x_{21} \\ x_{31} \\ x_{02} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{03} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} \\
&= \sum_{i=0}^{3} \sum_{j=0}^{3} u^i (0)^j C_{ij} = \sum_{i=0}^{3} u^i C_{i0} = u^T C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
\end{align*} \]

Similarly, \( v = \frac{1}{3} \)

\[ x(u, \frac{1}{3}) = \sum_{i=0}^{3} \sum_{j=0}^{3} u^i \left( \frac{1}{3} \right)^j C_{ij} = U^T C \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ (\frac{1}{3})^2 \end{pmatrix} = U^T C \mathbf{v}(\frac{1}{3}) \]
\[ u^T M_I X = u^T C A^T \]

\[ M_I X = C A^T \]

\[ \Rightarrow C = M_I X A^{-T} = M_I X M_I^T \]

\[ C = M_I X M_I^T \]

\[ x(u, v) = u^T C x \]

\[ = (u^T M_I) x (M_I^T v) \]

\[ = (M_I^T u)^T x (M_I^T v) \]

\[ = b(u)^T x b(v) \]

\[ = \sum_{i,j} b_i(u) x_i, b_j(v) \]

\[ b_i(u) b_j(v) \quad \text{Blending patch} \]