CS230 : Computer Graphics

Lecture 10: Rotations

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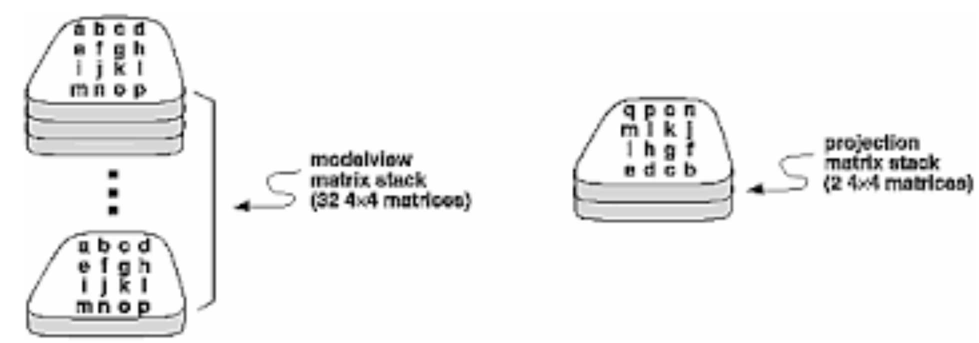
OpenGL Matrix Stacks

- OpenGL has modelview, projection, and texture matrix stacks
- allows for convenient management of hierarchical transformations
- stacks initialized with identity matrix




```
glMatrixMode
glPushMatrix
glPopMatrix
glLoadMatrix
glMultMatrix
glLoadIdentity
```

OpenGL Transformations

- Multiply the current matrix by the associated transformation matrix and
- replace the current matrix by the resulting product

```
glTranslate
glRotate
glScale

glOrtho
glFrustum
```

OpenGL Drawing

```
glBegin(GL_TRIANGLES);
glVertex2f(0.25, 0.25);
glVertex2f(0.75, 0.25);
glVertex2f(0.75, 0.75);
glEnd();
```

general rotations

Rotation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Z axis

The rows and columns are orthonormal

Rotation about an arbitrary axis

Rotating about an axis by theta degrees

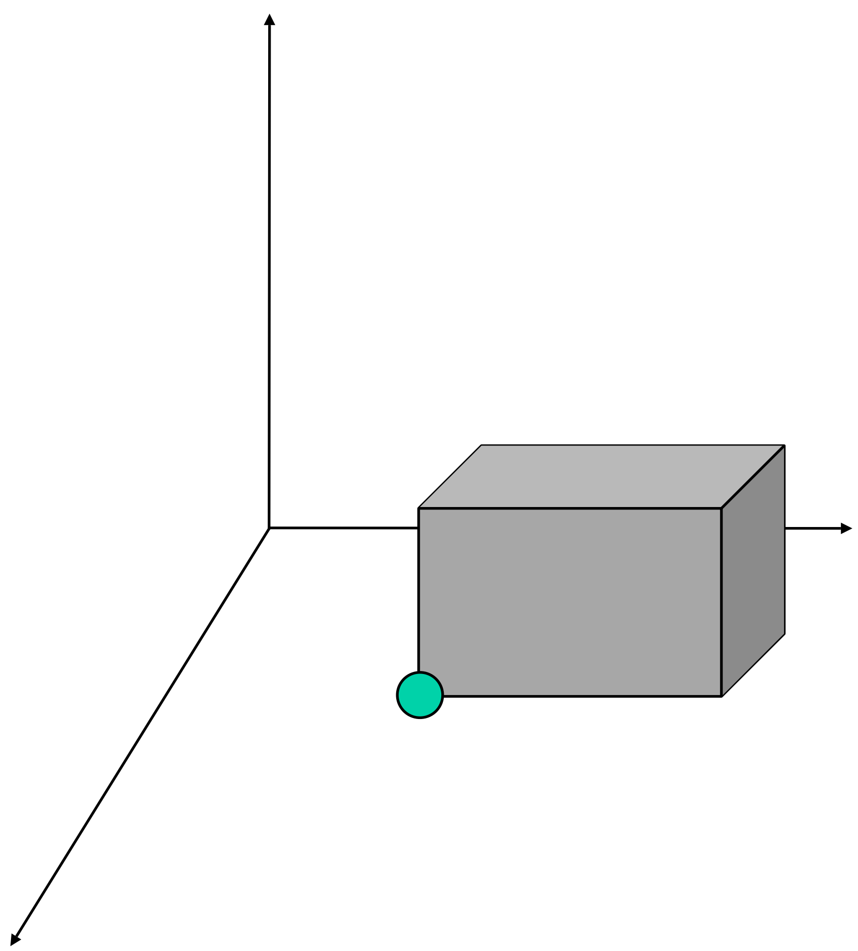
- Rotate about x to bring axis to xz plane
- Rotate about y to align axis with z -axis
- Rotate theta degrees about z
- Unrotate about y, unrotate about x

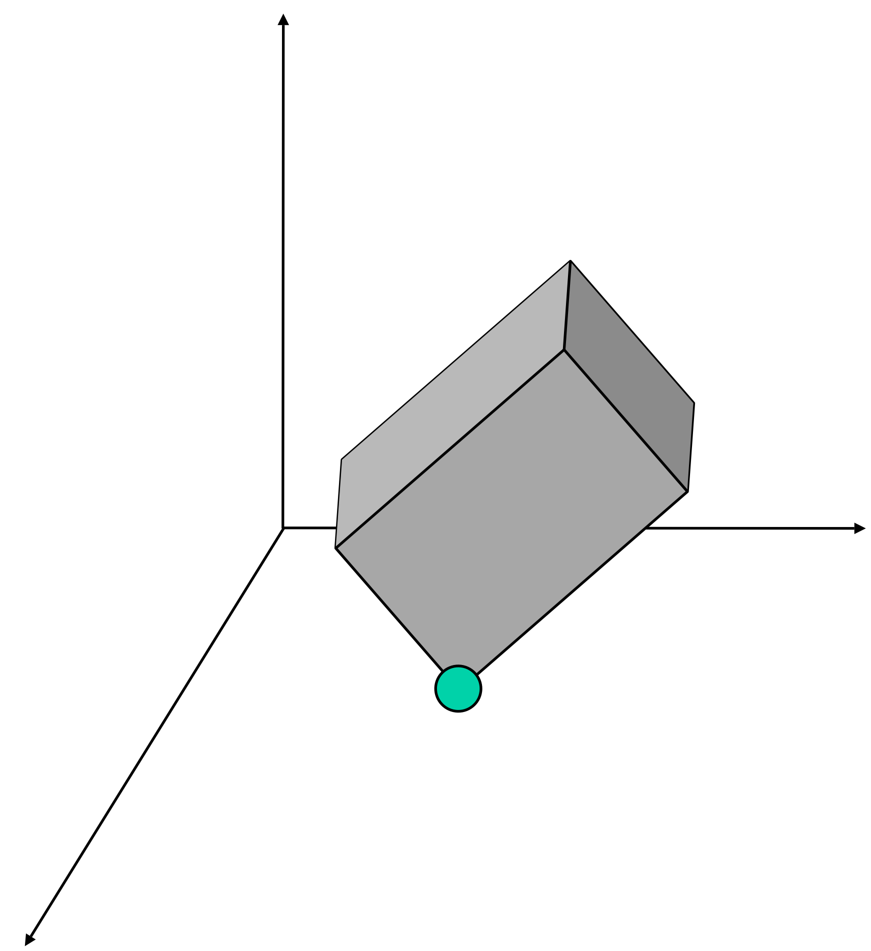
$$\mathbf{M} = \mathbf{Rx}^{-1}\ \mathbf{Ry}^{-1}\ \mathbf{Rz}(\theta)\ \mathbf{Ry}\ \mathbf{Rx}$$

- Can you determine the values of Rx and Ry?

Composite Transformations

- Rotating about a fixed point
  - **basic** rotation alone will rotate about origin but we want:

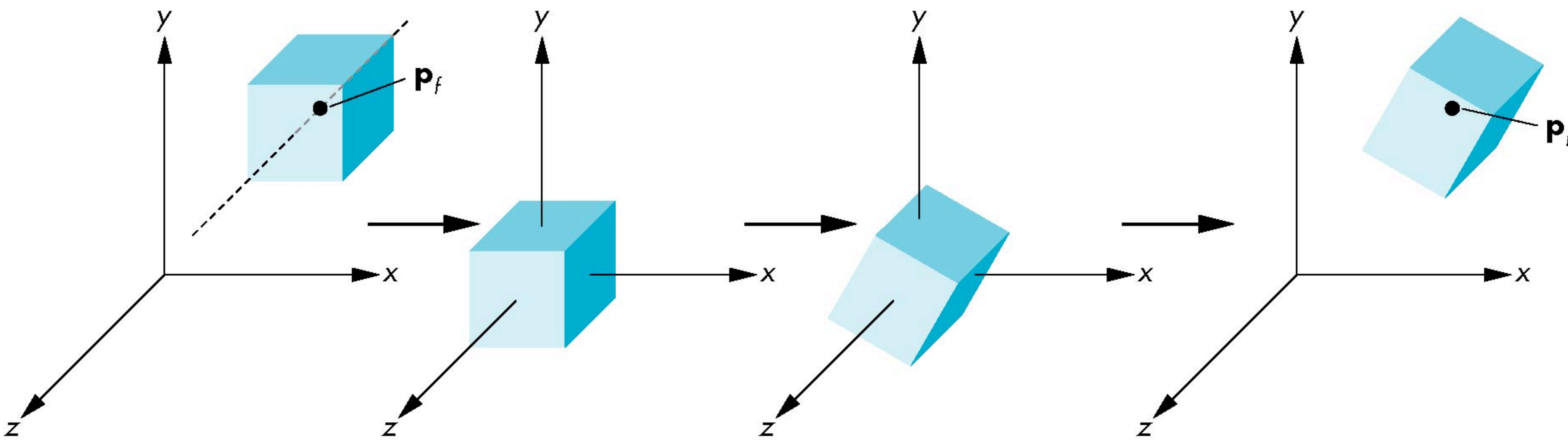




Composite Transformations

- Rotating about a fixed point
- Move fixed point (px,py,pz) to origin
- Rotate by desired amount
- Move fixed point back to original position

$$\mathbf{M} = \mathbf{T}(px, py, pz)\, \mathbf{R_z}(\theta)\, \mathbf{T}(-px, -py, -pz)$$



euler angles

Euler Angles

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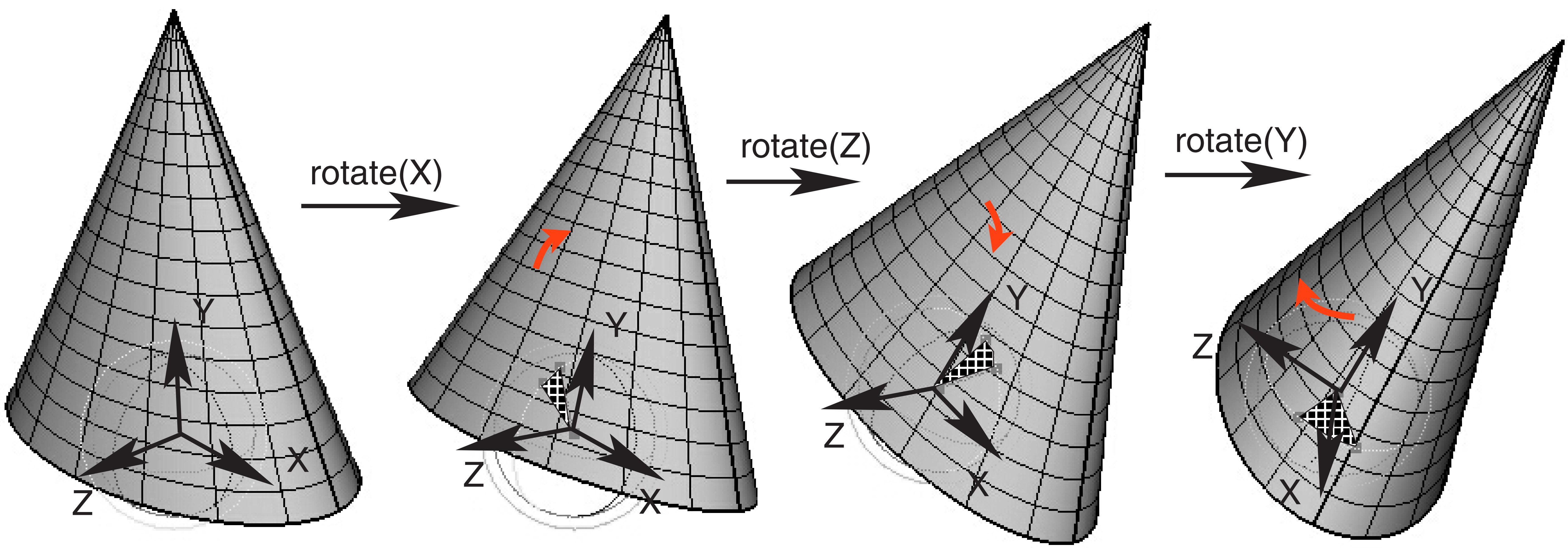
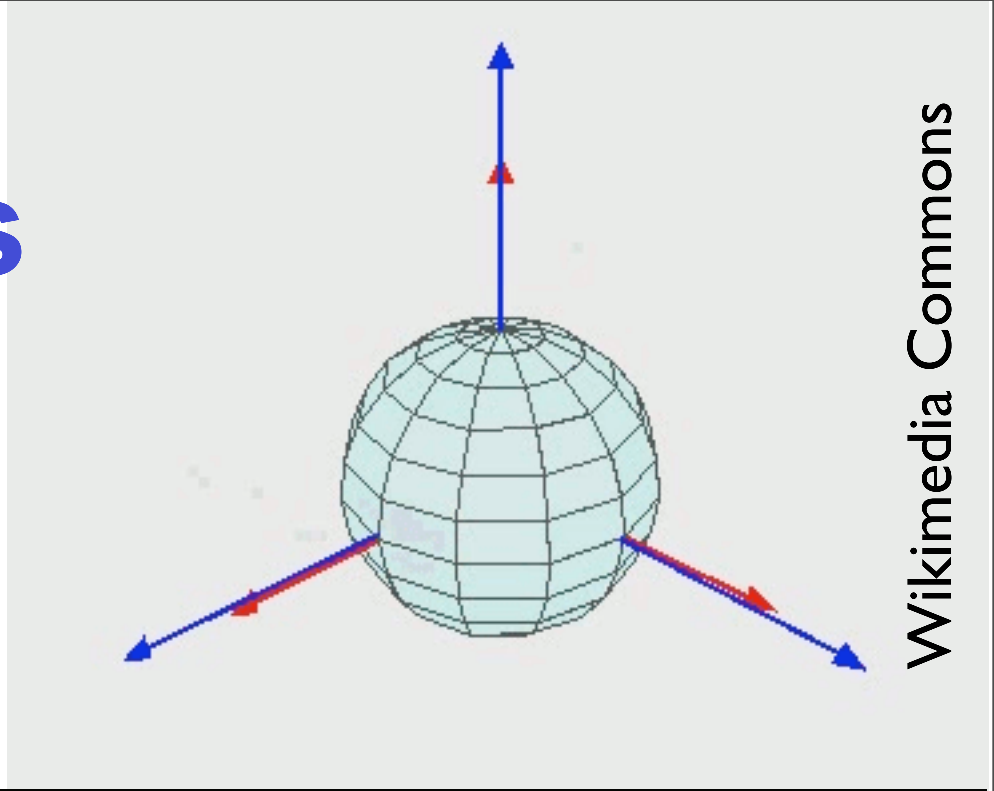
3 Euler angles can be used to specify an arbitrary orientation



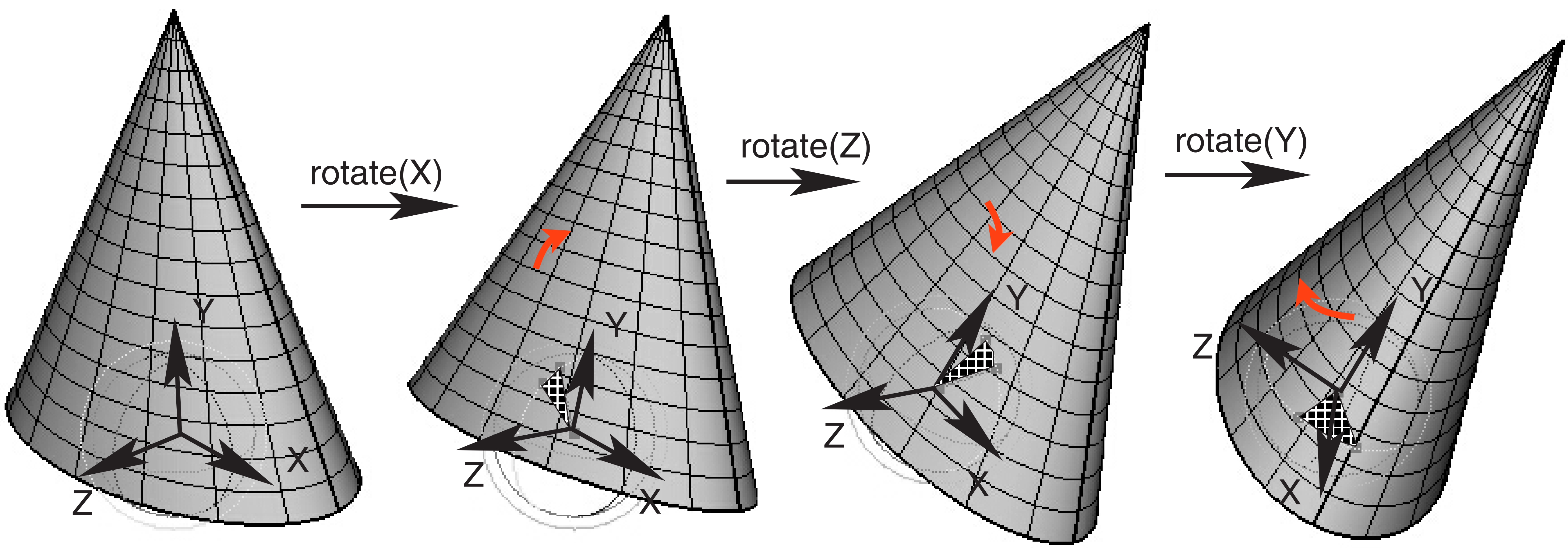
Shirley and Marschner

Three Euler angles can be used to specify arbitrary object orientation through a sequence of three rotations around coordinate axes embedded into the object

Euler Angles



3 Euler angles can be used to specify an arbitrary orientation

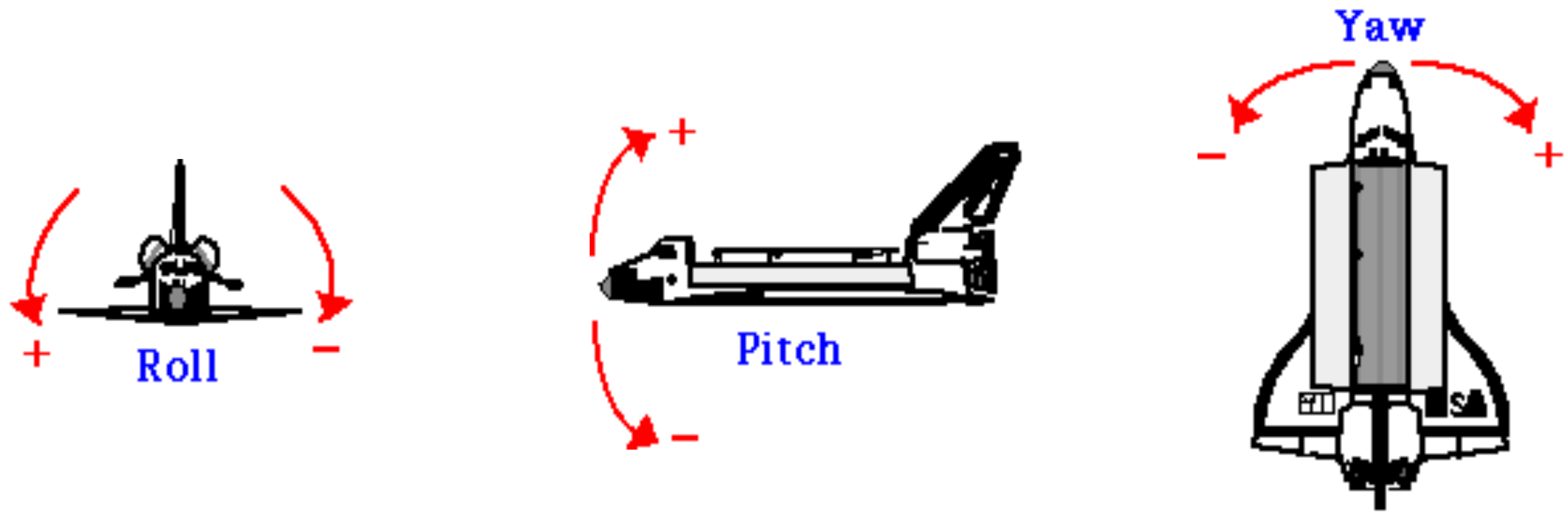


Shirley and Marschner

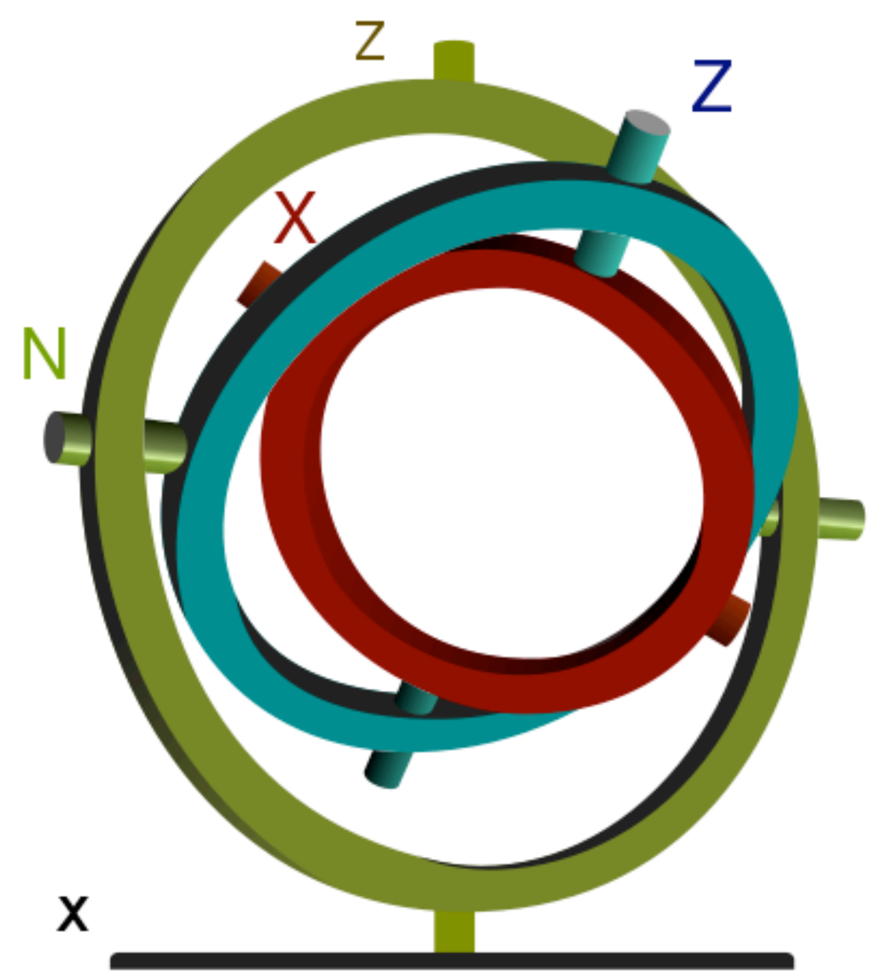
Three Euler angles can be used to specify arbitrary object orientation through a sequence of three rotations around coordinate axes embedded into the object

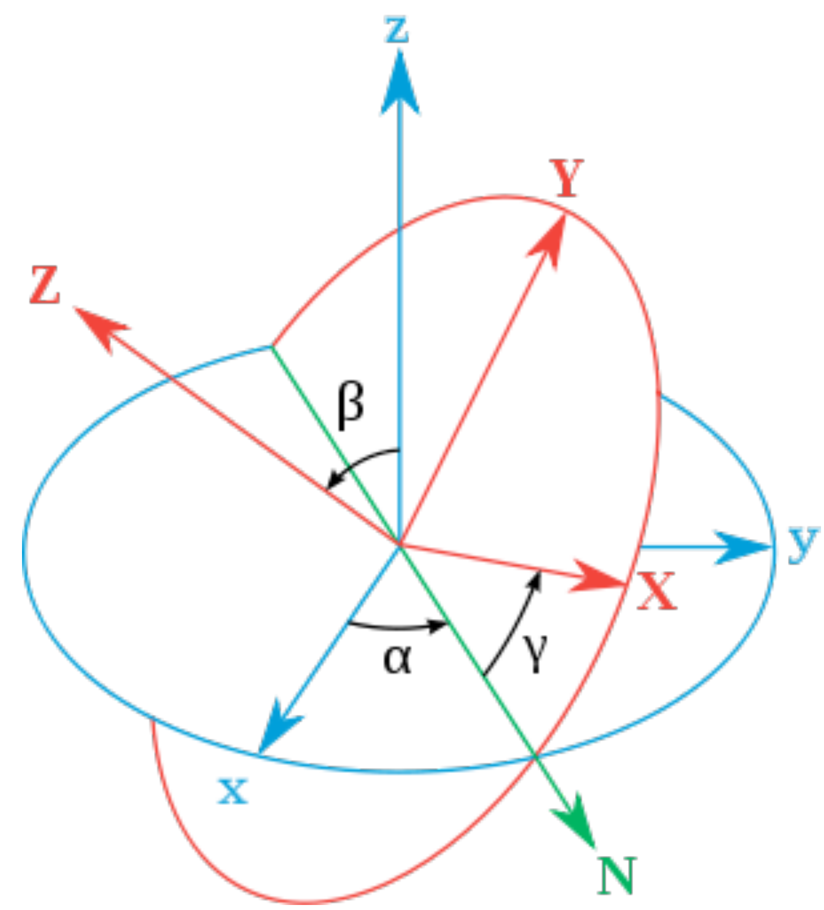
Euler Angles

- A general rotation is a combination of three elementary rotations:
  - around the x-axis (roll)
  - around the y-axis (pitch)
  - around the z-axis (yaw)



Gimbal and Euler Angles





Z-X'-Z''

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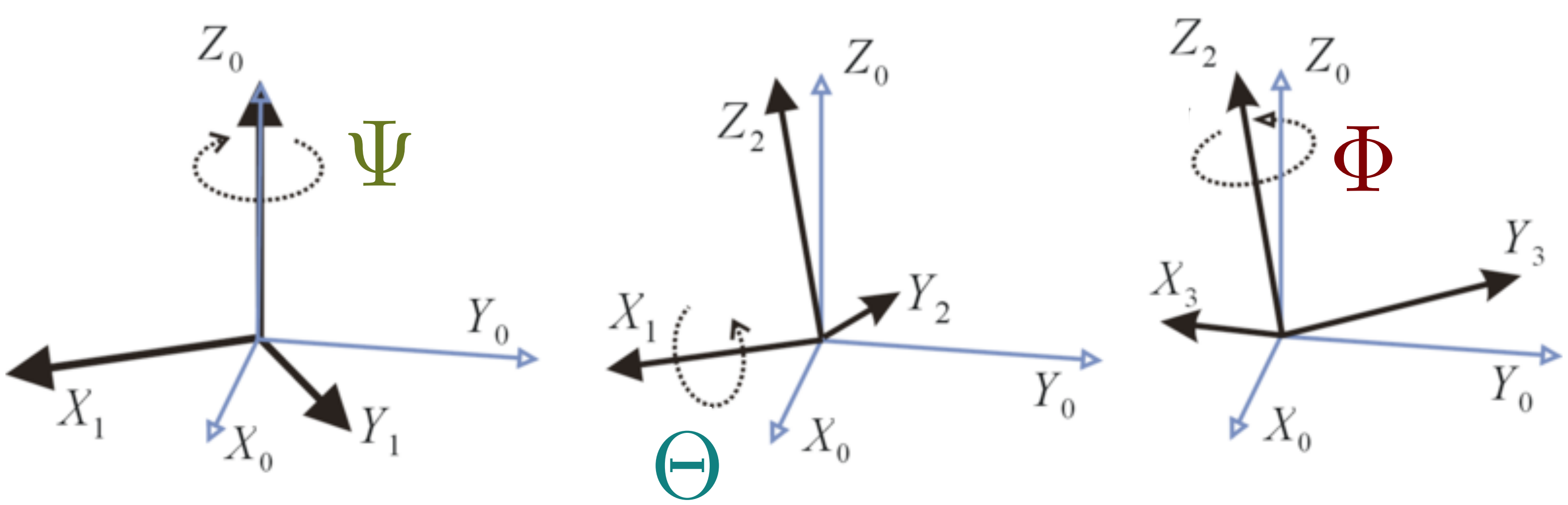
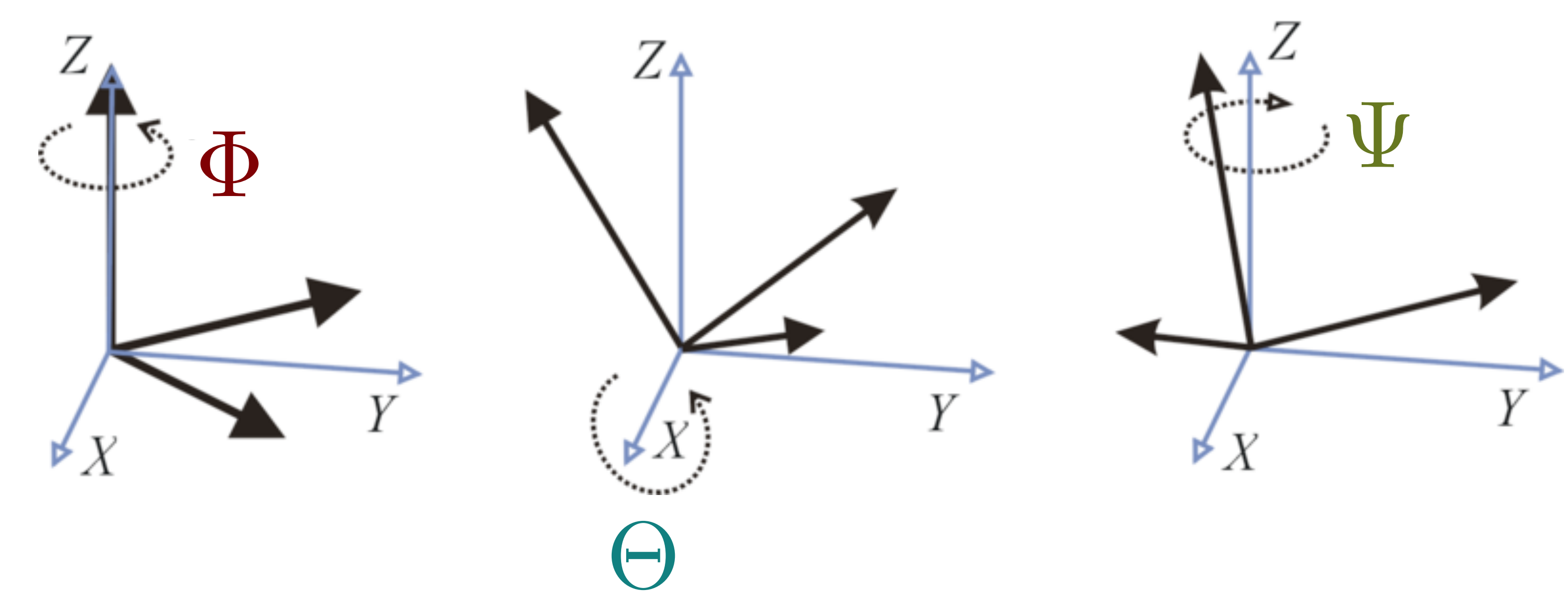
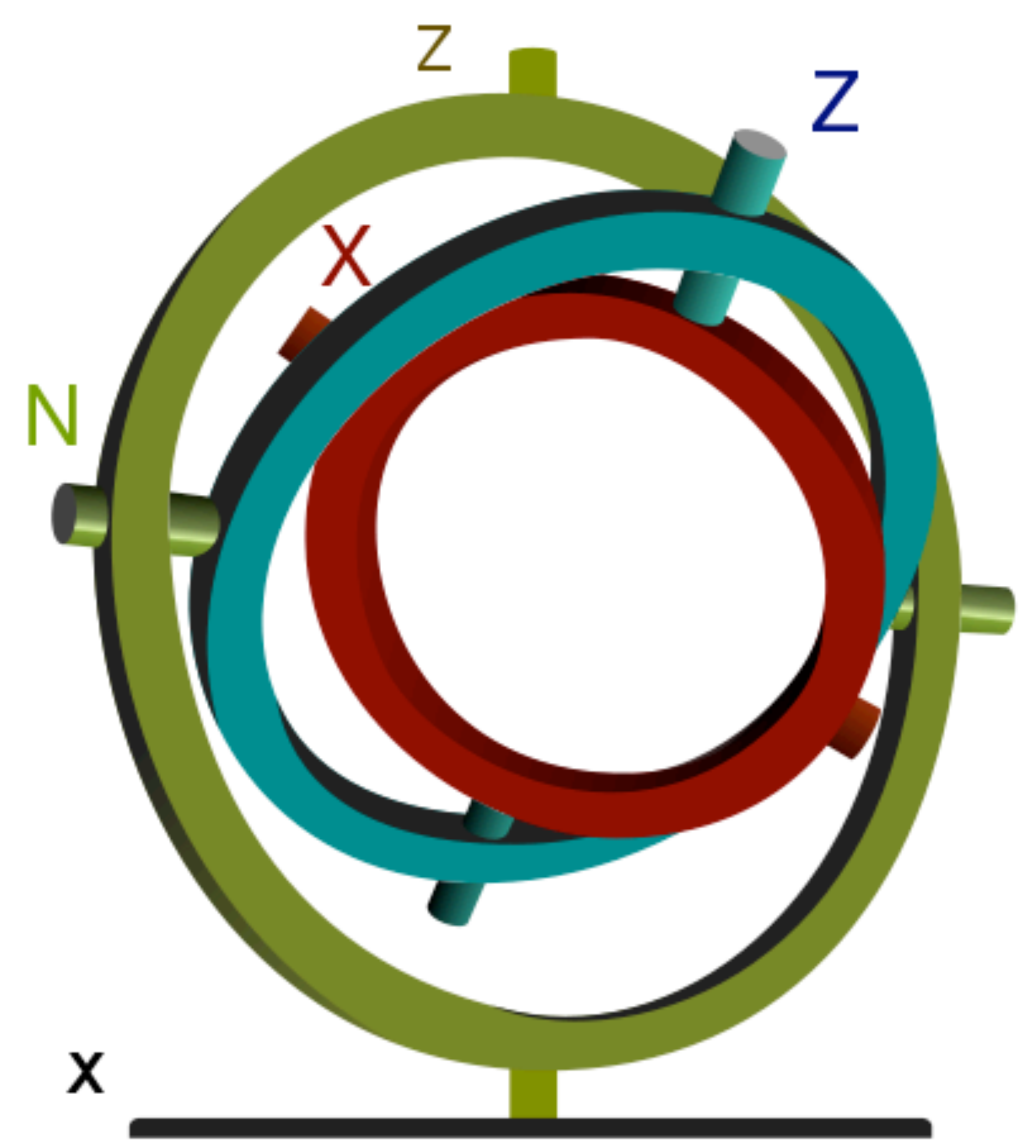
intrinsic



extrinsic





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intrinsic – rotations about the object fixed axes
 – first rotate outer gimbal, then middle gimbal, finally inner gimbal
extrinsic – rotations about the reference axes
 – first rotate inner gimbal, then middle gimbal, finally outer gimbal

Euler Angles and Rotation Matrices

$$\text{x-roll}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{y-pitch}(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{z-yaw}(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{R}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} c_2 c_3 & -c_2 s_3 & s_2 & 0 \\ s_1 s_2 c_3 + c_1 s_3 & -s_1 s_2 s_3 + c_1 c_3 & -s_1 c_2 & 0 \\ -c1 s_2 c_3 + s_1 s_3 & c_1 s_2 s_3 + s_1 c_3 & c_1 c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

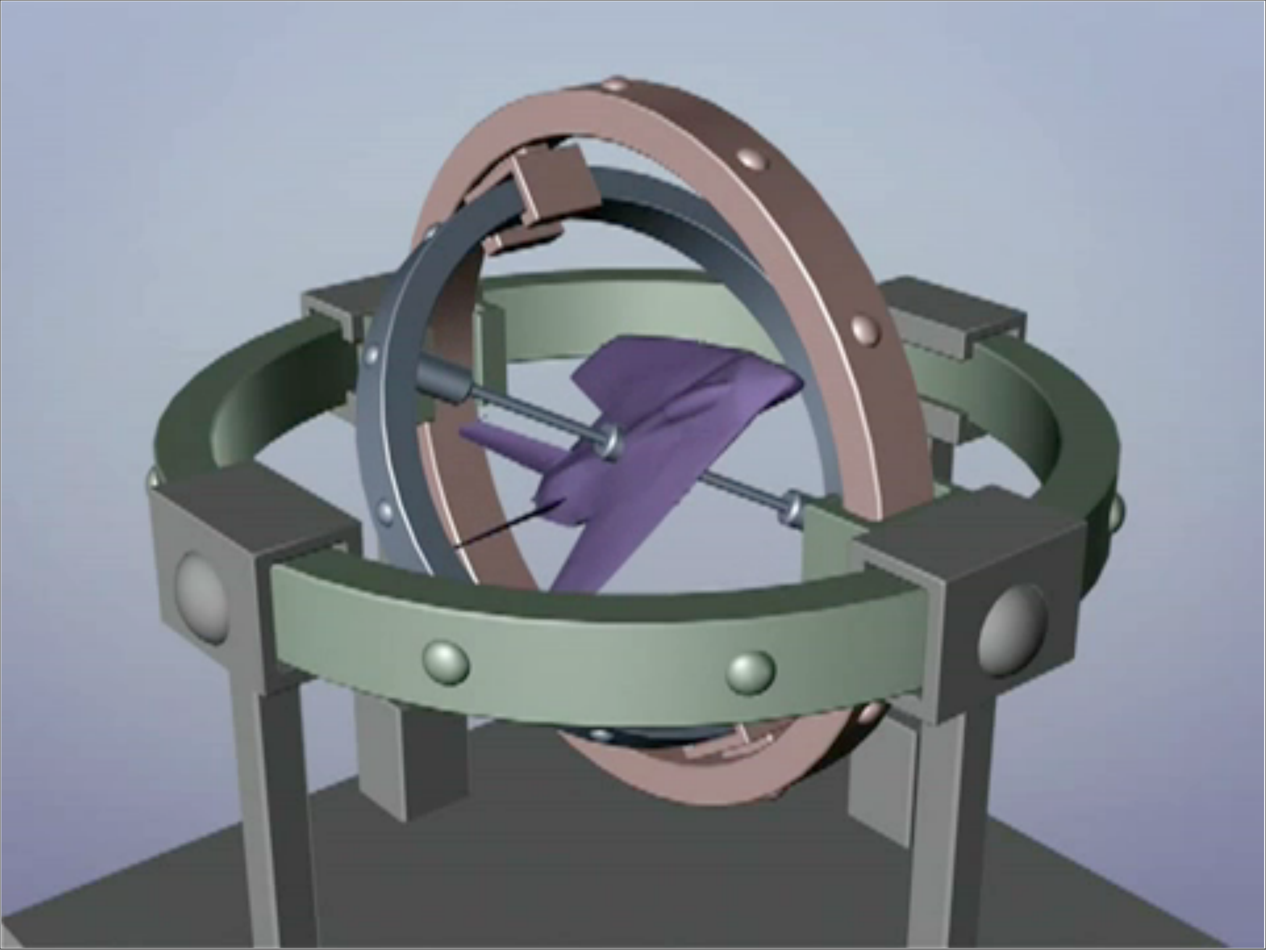
Interpolating Rotations

- naive element-by-element interpolation of rotation matrices does not work! E.g.,

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

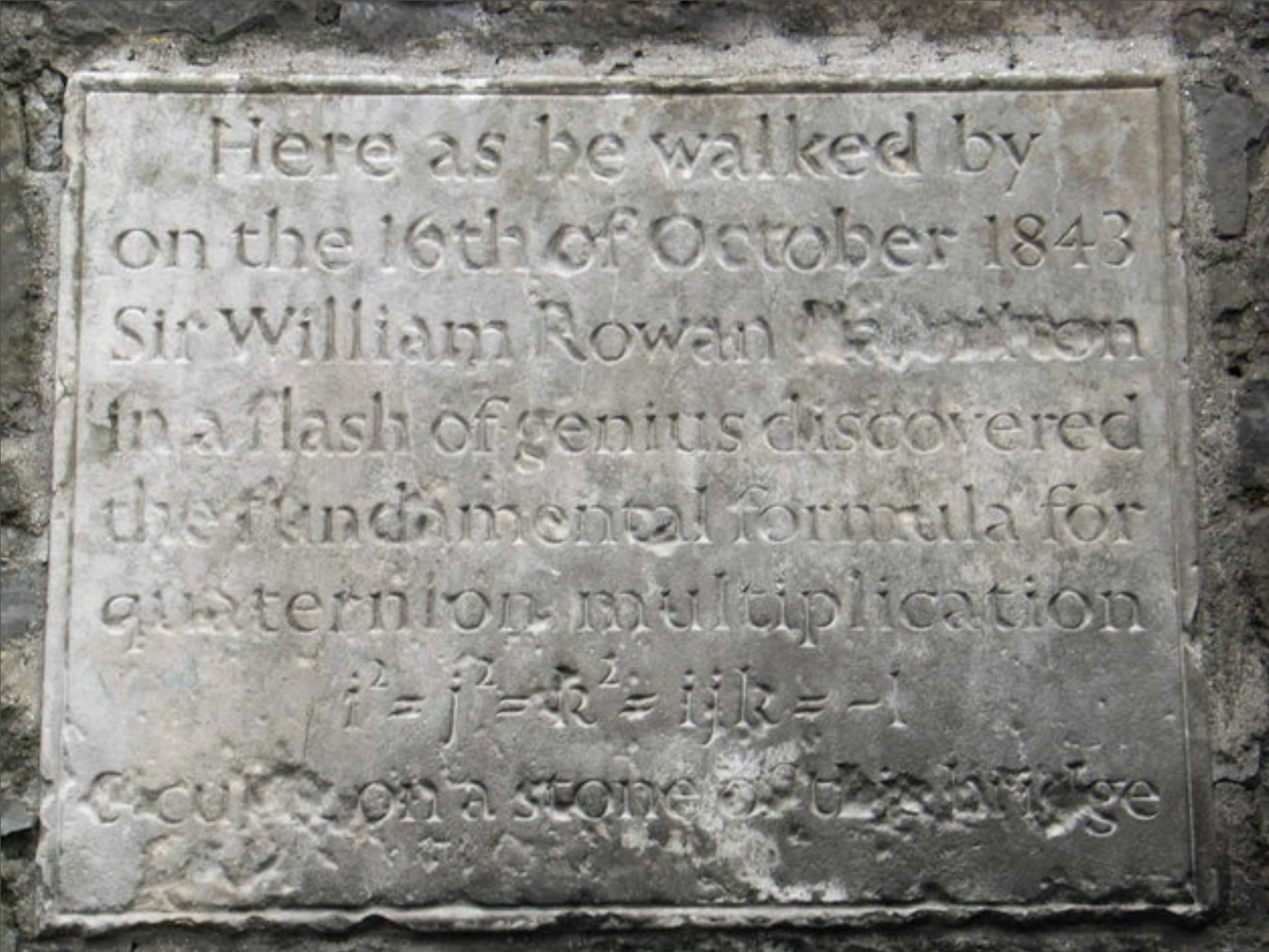
90 CW 90 CCW

- We can interpolate Euler angles, but there is a problem known as Gimbal Lock http://www.youtube.com/watch?v=zc8b2Jo7mno



http://www.youtube.com/watch?v=zc8b2Jo7mno

quaternions

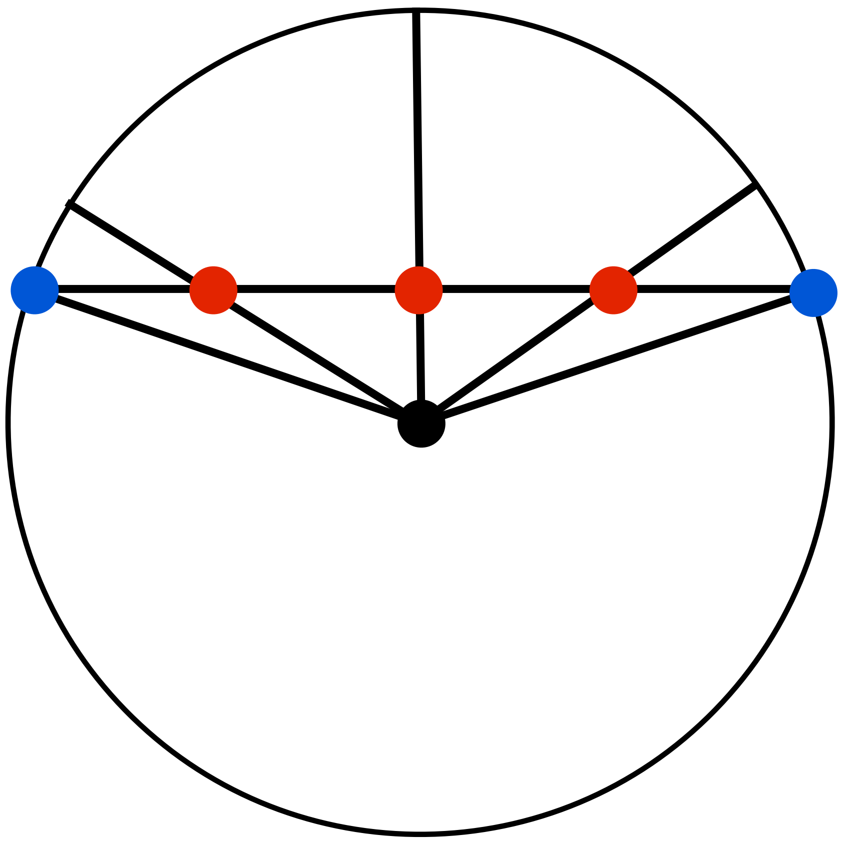

Here as he walked by
on the 16th of October 1843
Sir William Rowan
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
on a stone bridge

Quaternions

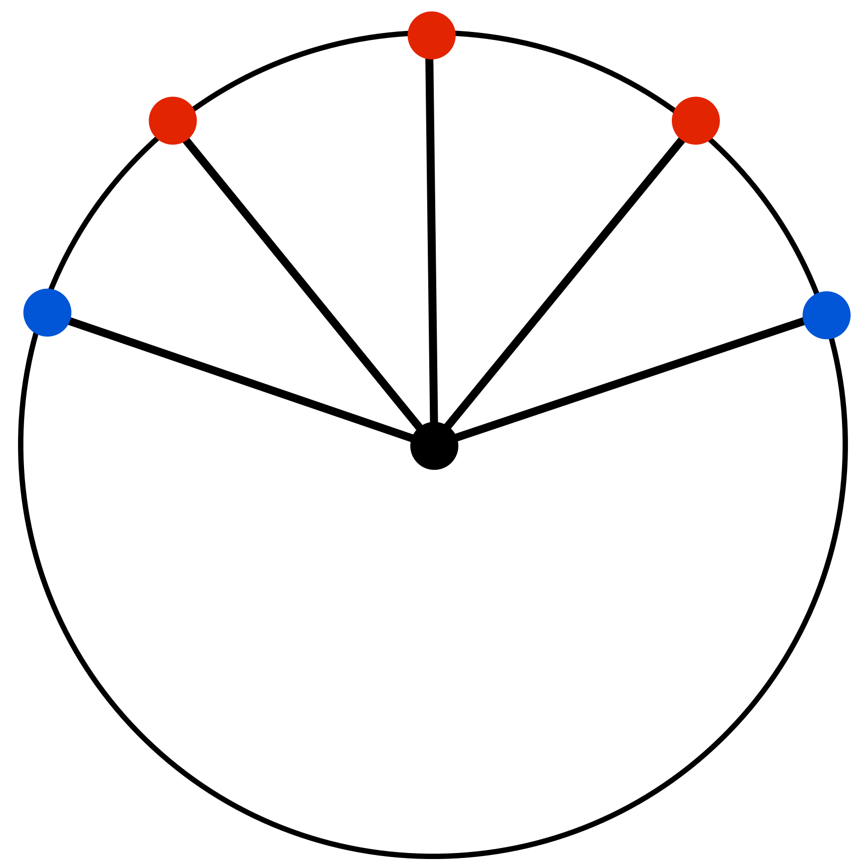
- axis/angle representation
- interpolates smoothly
- easy to compose

<whiteboard>

Quaternion Interpolation



linear



spherical linear
“slerp”

linear: treat quaternions as 4-vectors, note non-uniform speed
spherical linear: constant speed

Matrix form

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy & 0 \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx & 0 \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in Reality

- It's easiest to express rotations in Euler angles or Axis/angle

- We can convert to/from any of these representations

- Choose the best representation for the task
  - input:Euler angles
  - interpolation: quaternions
  - composing rotations: quaternions, orientation matrix