Linear Systems

1. For each of the following statements, indicate whether the statement is true or false.
   
   **T/F** If a matrix $A$ is singular, then the number of solutions to the linear system $Ax = b$ depends on the particular choice of right-hand-side $b$.
   
   **T/F** If a matrix $A$ is nonsingular, then the number of solutions to the linear system $Ax = b$ depends on the particular choice of right-hand-side $b$.
   
   **T/F** If a matrix has a very small determinant, then the matrix is nearly singular.
   
   **T/F** If any matrix has a zero on its main diagonal, then it is necessarily singular.

2. Can a system of linear equations $Ax = b$ have exactly two solutions? Explain your answer.

**LU Factorization and Gaussian Elimination**

3. For each of the following statements, indicate whether the statement is true or false.
   
   **T/F** If a triangular matrix has a zero on its main diagonal, then it is necessarily singular.
   
   **T/F** The product of two upper triangular matrices is upper triangular.
   
   **T/F** If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.
   
   **T/F** Once the LU factorization of a matrix has been computed to solve a linear system, then subsequent linear systems with the same matrix but different right-hand-side vectors can be solved without refactoring the matrix.

4. (T&B 20.2) Suppose $A \in \mathbb{R}^{n \times n}$ has an $LU$ factorization. Suppose that $A$ is banded with bandwidth $2p + 1$, i.e., $a_{ij} = 0$ for $|i - j| > p$. What can you say about the sparsity patterns of the factors $L$ and $U$ of $A$? Explain.

5. Consider $LU$ factorization with partial pivoting of the matrix $A$ which computes
   \[ M_{n-1}P_{n-1} \cdots M_3P_3M_2P_2M_1P_1A = U \]
   
   where $P_i$ is a row permutation matrix interchanging rows $i$ and $j > i$. 
(a) Show that the matrix $P_3P_2M_1P_2^{-1}P_3^{-1}$ has the same structure as the matrix $M_1$.
(b) Explain how the above expression is transformed into the form $PA = LU$, where $P$ is a row permutation matrix.

**Cholesky Factorization**

6. (Heath 2.37) Suppose that the symmetric $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} \alpha & \mathbf{a}^T \\ \mathbf{a} & A \end{pmatrix}$$

is positive definite.

(a) Show that the scalar $\alpha$ must be positive and the $n \times n$ matrix $A$ must be positive definite.
(b) What is the Cholesky factorization of $B$ in terms of $\alpha$, $\mathbf{a}$, and the Cholesky factorization of $A$?

**Singular Value Decomposition**

7. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

8. Let $A$ be an $m \times n$ singular matrix of rank $r$ with SVD

$$A = U \Sigma V^T = \begin{pmatrix} u_1 & u_2 & \ldots & u_m \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first $r$ columns of $U$, $\tilde{U}$ consists of the remaining $m - r$ columns of $U$, $\hat{V}$ consists of the first $r$ columns of $V$, and $\tilde{V}$ consists of the remaining $n - r$ columns of $V$. Give bases for the spaces range($A$), null($A$), range($A^T$) and null($A^T$) in terms of the components of the SVD of $A$, and a brief justification.

9. Show that for an $m \times n$ matrix of full column rank $n$, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range($A$). Hint: use the SVD of $A$. 