Homework 4 Solutions
CS 210

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Eigenvalue problems

1. Consider the following statements about eigenvalue problems. Mark each statement as true or false.

T / F A defective eigenvalue is one where the geometric multiplicity is greater the algebraic multiplicity.
T / F A good way to compute eigenvalues is by finding roots of the associated characteristic polynomial.
T / F An orthogonal projection matrix has one eigenvalue equal to 0 and the other eigenvalues equal to 1.
T / F Symmetric matrices have orthogonal set of eigenvectors.
T / F A projection matrix must have at least one eigenvalue equal to 0.
T / F A matrix that has an orthogonal set of eigenvectors can be decomposed as $A = UΛU^T$ where $U$ is orthogonal and $Λ$ is diagonal.

Notes: (c) and (e) are false because a projection (orthogonal or not) might have more or less than one eigenvalue equal to zero. Take as examples the $n \times n$ ($n > 2$) identity matrix (orthogonal projection onto the whole space) and zero matrix (orthogonal projection onto the zero-dimensional subspace).

2. (Heath 4.26) Which of the following conditions necessarily imply that an $n \times n$ real matrix $A$ is diagonalizable (i.e., similar to a diagonal matrix)?

(a) $A$ has $n$ distinct eigenvalues.
(b) $A$ has only real eigenvalues.
(c) $A$ is nonsingular.
(d) $A$ is equal to its transpose.
(e) $A$ commutes with its transpose.
Notes: A simple matrix \(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\) provides a counterexample to (b) and (c). It has an eigenvalue 2 with algebraic multiplicity 2. However, it is already in Jordan normal form, and indicates the eigenvalue 2 is defective, and hence the matrix is not diagonalizable. Since the matrix is triangular, its determinant is the product of the diagonal entries, and is 4. Hence the matrix is nonsingular. (e) is the definition of a normal matrix. Real normal matrices are orthogonally diagonalizable. (e) implies (d) (a symmetric matrix is a normal matrix).

3. (Heath 4.42)
   (a) If a matrix \(A\) has a simple dominant eigenvalue \(\lambda_1\), what quantity determines the convergence rate of the power method for computing \(\lambda_1\)?
   (b) How can the convergence rate of power iteration be improved?

4. (Heath 4.3) Given an approximate eigenvector \(x\) of \(A\), what is the best estimate (in the least squares sense) for the corresponding eigenvalue?

5. (Heath 4.1)
   (a) Prove that 5 is an eigenvalue of the matrix
   \[
   A = \begin{pmatrix}
   6 & 3 & 3 & 1 \\
   0 & 7 & 4 & 5 \\
   0 & 0 & 5 & 4 \\
   0 & 0 & 0 & 8
   \end{pmatrix}
   \]
   (b) Exhibit an eigenvector of \(A\) corresponding to the eigenvalue 5.

6. (Heath 4.24) Let \(A\) be an \(n \times n\) real matrix of rank one.
   (a) Show that \(A = uv^T\) for some nonzero real vectors \(u\) and \(v\).
   (b) Show that \(u^T v\) is an eigenvalue of \(A\).
   (c) If power iteration is applied to \(A\), how many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue?

Nonlinear Equations

7. Consider the following statements about nonlinear equation solving. Mark each statement as true or false.

   T / F A small residual \(||f(x)|||\) guarantees an accurate solution of a system of nonlinear equations \(f(x) = 0\).
   False: the conditioning of the problem may mean that our solution is inaccurate even with a small residual.

   T / F Newton’s method is an example of a fixed point iteration.
   F: Newton’s method is a root-finding method, not a fixed point iteration.

   T / F If an iterative method for solving a nonlinear equation gains more than one bit of accuracy per iteration, then it is said to have a superlinear convergence rate.
   False: In a method with linear convergence, the number of bits of accuracy gained per iteration is \(\approx \log_\beta(C)\) where \(\beta\) is the base of the number system and \(C\) is the convergence constant.

   T / F Newton’s method always converges quadratically.
   False: This is not true in the case of multiple roots.

   T / F The nonlinear root-finding problem \(f(x) = 0\) has either zero, one, or infinitely many solutions.
   False: Other numbers of roots are also possible for a nonlinear equation.
A fixed point of a function \( f(x) \) is a point \( x^* \) such that \( f(x^*) = 0 \).

False: \( f(x^*) = 0 \) then \( x^* \) is a root of \( f \). A fixed point \( x^* \) satisfies \( f(x^*) = x^* \).

8. Compare Newton’s method and the Secant Method for solving a scalar nonlinear equation. What are the advantages and disadvantages of each?

**Answer:**

Newton’s method uses the iterative scheme:

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.
\]

The Secant method is similar, but replaces the derivative \( f'(x_k) \) with the approximation \( f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \), giving the iterative scheme

\[
x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}.
\]

The advantage of Newton’s method over the Secant method is that it has locally quadratic convergence (for a simple root). The Secant method has a convergence rate of \( \sim 1.618 \). The advantage of the Secant method is that it does not require an expression for \( f'(x) \) and does not require evaluation of \( f'(x) \). Only one new function evaluation is computed per iteration in the Secant method, whereas Newton’s method requires two function evaluations. In practice, although the Secant method requires more iterations, the savings in cost per iteration may make the Secant method the more efficient choice.

9. (Heath 5.1) Consider the nonlinear equation

\[ f(x) = x^2 - 2 = 0. \]

(a) With \( x_0 = 1 \), as a starting point, what is the value of \( x_1 \) if you use Newton’s method for solving this problem?

(b) With \( x_0 = 1 \) and \( x_1 = 2 \) as a starting points, what is the value of \( x_2 \) if you use the secant method for the same problem?

**Answer:**

\[ f'(x) = 2x \]

(a) \( x_0 = 1 \)

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 + \frac{1}{2} = \frac{3}{2} \]

(b) \( x_0 = 1, x_1 = 2 \)

\[ x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = \frac{2 - 2(2 - 1)}{2 - (-1)} = \frac{2}{3} = \frac{4}{3} \]

10. (Heath 5.12) Newton’s method for solving a scalar nonlinear equation \( f(x) = 0 \) requires computation of the derivative of \( f \) at each iteration. Suppose that we instead replace the true derivative with a constant value \( d \), that is, we use the iteration scheme

\[ x_{k+1} = x_k - \frac{f(x)}{d}. \]

(a) Under what condition on the value of \( d \) will this scheme be locally convergent?
Answer: Let \( g(x) = x - \frac{f(x)}{d} \) so that \( g'(x) = 1 - \frac{f'(x)}{d} \). \( x^* \) is a solution which satisfies \( x^* = g(x^*) \).

The scheme will be locally convergent when \( |g'(x^*)| < 1 \).

\[
\begin{align*}
|g'(x^*)| &< 1 \\
|1 - \frac{f'(x^*)}{d}| &< 1 \\
0 &< \frac{f'(x^*)}{d} < 2 \\
0 &< f'(x^*) < 2d
\end{align*}
\]

(b) What will be the convergence rate, in general?

Answer: In class it was shown that

\[
e^{k+1} e^k = g'(x^*) + \frac{e^k}{2} g''(x^*) + O(e^{2k})
\]

When we take the limit \( \lim_{k \to \infty} \frac{e^{k+1}}{e^k} \) we are left with the constant \( C = g'(x^*) \) so the general convergence rate is linear.

(c) Is there any value of \( d \) that would still yield quadratic convergence?

Answer: If we set \( d = f'(x^*) \), \( g'(x^*) = 0 \) and the linear term disappears. Taking the limit \( \lim_{k \to \infty} \frac{e^{k+1}}{e^k} \) we are left with the constant \( C = \frac{g''(x^*)}{2} \) so we have quadratic convergence.

11. (Heath 5.10) Carry out one iteration of Newton’s method applied to the system of nonlinear equations

\[
\begin{align*}
x_1^2 - x_2^2 &= 0 \\
x_1 x_2 &= 1
\end{align*}
\]

with starting value \( x_0 = (0, 1)^T \).

Answer:

\[
f(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1 x_2 - 1 \end{pmatrix}
\]

\[
J_f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}
\]

To carry out one iteration we must solve the following linear system

\[
J_f(x_0)s_0 = -f(x_0)
\]

\[
\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} s_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
s_0 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\]

We can then carry out the update as

\[
x_1 = x_0 + s_0 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\]

12. Computer problem (Heath 5.3) Implement the bisection, Newton, and secant methods for solving nonlinear equations in one dimension, and test your implementation by finding at least one root for each of the following equations. What termination criterion should you use? What convergence rate is achieved in each case?
(a) \( x^3 - 2x - 5 = 0 \).
(b) \( e^{-x} = x \).
(c) \( x \sin(x) = 1 \).
(d) \( x^3 - 3x^2 + 3x - 1 = 0 \).

Solution Outline:
- implement Newton’s method, secant method, and bisection method and include your code - give one root for each method for each of parts a-d, e.g., a table
  — NM — S — B
  a
  b
  c
  d
  with the approximate roots.
  - discuss termination criteria for each algorithm with some justification. For example, using a tolerance on the change in \( x \) in subsequent iterations, evaluating with how close \( f(x) \) is to zero, or a combination of these two ideas.
  - give a table for the convergence rates achieved for the roots in the first table, so another table of this form but this time with the rates:
    — NM — S — B
    a
    b
    c
    d
  - for one example, show the convergence, e.g., choosing NM and c, show
    \( x_0 \)
    \( x_1 \)
    \( x_2 \)
    \( x_3 \)
    ...
    \( x_n \)
    and compute the convergence rate using the formula \( \lim_{k \to \infty} \frac{|x_{k+1} - L|}{x_k - L} \).

Note: The above is only an outline; we will be flexible in grading since the question left some details vague.