Iterative Methods (Heath § 11.5)

- direct methods compute exact solution subject to rounding error
- can be too costly in storage & work for large systems
- iterative methods start w/ guess x_0 & improve until desired accuracy is achieved

Several approaches based on fixed point iteration

$$x_{k+1} = G x_k + c$$

Matrix G can be obtained by splitting

$$A = M - N$$

$$\boxed{M x_{k+1} = N x_k + b}$$

$$x_{k+1} = M^{-1} N x_k + M^{-1} b$$

$$G(x) = M^{-1} N$$

convergent if $\rho(M^{-1}N) < 1$

M is ~~easier~~ chosen so that it is easier to solve than A.

Jacobi Method

$$M = D$$

$$N = -(L+U)$$

$$Dx^{(k+1)} = -(L+U)x^{(k)} + b$$

$$x^{(k+1)} = D^{-1}(b-(L+U)x^{(k)})$$

Equation for a single component i

$$x_i^{(k+1)} = \frac{b_i - \sum_{j\neq i} a_{ij}\, x_j^{(k)}}{a_{ii}}$$

- slow convergence usually
- easy to parallelize.

Gauss-Seidel

- use updated values of $x_i^{(k+1)}$ after they've been computed:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j<i} a_{ij}\, x_j^{(k+1)} - \sum_{j>i} a_{ij}\, x_j^{(k)}\right]$$

- equivalent to

$$(D+L)x^{(k+1)} = -Ux^{(k)} + b$$

Gauss-Seidel (cont.)

\+ faster convergence than Jacobi

\- sequential evaluation ~~etc~~ of $x_i^{(k+1)}$

\+ in-place update of x_i

Preconditioning

- improve condition # of a matrix

$M \approx A$, easier to solve M

replace

$$Ax = b$$

with

$$M^{-1}Ax = M^{-1}b$$

Note that "M^{-1}" does not necessarily mean we compute the inverse of M.

~~Ex~~ Intuitively: stretch quadratic form to make it more spherical.

$$M = EE^T \implies M^{-1} = E^{-T}E^{-1}$$

$$M^{-1}Ax = M^{-1}b$$

$$E^{-T}E^{-1}Ax = E^{-T}E^{-1}b$$

$$\implies \left[E^{-1}AE^{-T}\right]\left(E^{T}x\right) = E^{-1}b$$

$$\hat{A}\,\hat{x} = \hat{b}$$

Applying CG to the ~~a~~ transformed equation, ~~and~~ get ~~[illegible]~~ terms with E^{-1}. These can be eliminated to get the following PCG algorithm:

$r_0 = b - Ax_0$

$s_0 = M^{-1}r_0$

for $k = 0, 1, 2, \ldots$

$$\alpha_k = \frac{r_k^T M^{-1} r_k}{s_k^T A s_k}$$

$$x_{k+1} = x_k + \alpha_k s_k$$

$$r_{k+1} = r_k - \alpha_k A s_k$$

$$\beta_{k+1} = \frac{r_{k+1}^T M^{-1} r_{k+1}}{r_k^T M^{-1} r_k}$$

$$s_{k+1} = M^{-1}r_{k+1} + \beta_{k+1} s_k$$