Computations involving **continuous** mathematics
- math arising in science and engineering
- e.g., simulation
- financial models
- optimization problems
- design and analysis of algorithms
- most problems can't be solved exactly
- find some iterative process that converges to the solution and cut it off at some point
- seek rapid convergence
- assess error

Simulation:
- solve problems that can't otherwise be solved
- explore parameter space more economically
- no "build-and-test"
- "virtual prototyping"

1. "Mathematical model" - Applied Math
2. Equations
3. Algorithms to solve equations numerically
4. Implement
5. Run
6. Visualize or otherwise evaluate the result
7. Interpret and validate
Well-posedness (vs. ill-posedness)
- solution exists
- unique
- depends continuously on data

In numerical computations, small changes inevitable.
Problem may be well-posed but still sensitive
- measure of sensitivity of a problem = "condition number"

Stable algorithm - doesn't introduce sensitivities or ill-posedness to a well-posed problem.
(just because math. prob. well-posed, doesn't mean algorithm is ...). NaN's

1.1.2. General Strategy
infinite ← finite

- dim spaces
- integrals
- derivatives
- diff. eq.
- nonlinear
- high order
- complicated
- function
- general matrices

Approximations - w/ (arbitrary?) good accuracy.
Sources of Error

1. Modeling
   simplify the problem to facilitate studying it or some aspect of it.

2. Empirical Measurements
   instrument error
   human error
   system noise

3. Other errors in input data during computation ...

4. truncation error or discretization error
   turning infinite into finite
   cut off at some point or sample function with limited accuracy.

5. Rounding
   represent real \\#5 w/ finite precision

Ex. Approx. area of earth
   \[ A = 4\pi r^2 \]
   \[ r = \text{modeling error} \]

\[ \pi \approx 3.1415926 \ldots \]
\[ r \approx 6.370 \text{ km, approximate} \]
\[ 0.8376 \]
\[ 5,099,041,355 \]

Ex. 141
12.2 Absolute Error + Relative Error

- absolute error = approx. - exact

- relative error = \frac{\text{absolute error}}{\text{exact}} * \text{in practice use the approximate value.}

- Relative error can also be expressed as percentage
- Another useful interpretation: # of correct significant digits

\[
\text{rel. err} \approx 10^{-p} \\
\Rightarrow \sim p \text{ significant digits correct.}
\]

\[
\frac{1}{100} = 10^{-3}
\]

E.g. \[
\frac{100}{12345} \approx \frac{1}{100} = 10^{-2} \Rightarrow 2 \text{ sig. digits.}
\]

E.g. \[
\frac{12345 \text{ approx. soln.}}{12445}
\]
[2.3] Data Error + Computational Error.

Compute \( f(x) \), \( f: \mathbb{R} \rightarrow \mathbb{R} \)

\( x \) true input
\( f(x) \) true output

\( \hat{x} \) approx. input
\( \hat{f} \) approx. function evaluation

Total error = \( \hat{f}(\hat{x}) - f(x) \)
= \( \hat{f}(\hat{x}) - f(x) + (f(\hat{x}) - f(x)) \)
= \( \hat{f}(\hat{x}) - f(\hat{x}) \) \( + \) \( f(x) - f(x) \)

\[ \text{computational error} \]
\[ \text{propagated data error} \]

\[ \text{diff. of exact function on approx. function on same input} \]
\[ \text{choice of algorithm does not affect propagated data error.} \]

Ex. 1.2. Approximate \( \sin \left( \frac{\pi}{8} \right) \) \( \approx 0.3827 \)

\( T \approx \frac{\pi}{8} \)
\( x = \frac{\pi}{8} \)

\( \hat{x} = \frac{3}{8} \)

Truncate

Taylor series \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \)

\( \sin x \approx x \)

\( \sin \left( \frac{\pi}{8} \right) \approx \sin \left( \frac{3}{8} \right) \times \frac{3}{8} = 0.375 \)

Propagated data error \( \uparrow \)
Computational error \( \uparrow \)
\( f(\hat{x}) - f(x) \approx \cdot 375 - \cdot 3827 = -\cdot 0077 \)

\( f(\hat{x}) = \sin(\frac{x}{8}) \approx \cdot 3663 \)

propagated data error

\( f(\hat{x}) - f(x) = \cdot 3663 - \cdot 3827 = -\cdot 0164 \)

computational error

\( f(\hat{x}) - f(x) = \cdot 375 - \cdot 3663 = \cdot 0087 \)

Note: the two errors have opposite signs so partially offset each other, but could have same sign.

In what case above would each type of error dominate?

1.2.5 Truncation Error + Rounding Error

- **truncation error** — if we could use **exact arithmetic**, what would the result be
  - diff: = true result - algorithm
  - if exact arith.
  - on actual data

E.g., truncating infinite series
  - derivatives — finite diff.
  - terminating iterative sequence before convergence

\( f(\hat{x}) - f(x) \) Rounding Error.

diff between algo. result produced using **exact arith.** vs.
finite-precision rounded **arith.**
truncation error \[ f(x) - \hat{f}^{\text{exact}}(x) \]

rounding error \[ \hat{f}^{\text{exact}}(x) - \hat{f}^{\text{rounded}}(x) \]

\[ f(x) - \hat{f}^{\text{rounded}}(x) \]

**Ex. 1.3** Finite Difference Approximation

\[ f'(x) \approx \frac{f(x+h) - f(x)}{h} \]

Taylor's theorem

\[ f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots \]

\[ = f(x) + hf'(x) + \frac{h^2}{2} f''(e), \; e \in [x, x+h] \]

truncation error:

\[ \frac{f'(x) - f(x+h) - hf'(x)}{h} = \frac{f(x) - f(x+h) + \frac{h}{2} f''(e)}{h} \]

\[ = -\frac{1}{2} f''(e) \leq \frac{M}{2} h^2 \]

rounding error

rounding error in \( f \leq \varepsilon \Rightarrow \)

\[ \frac{|f'(x) - f(x+h) - hf'(x)|}{h} \leq \frac{1}{h} \left( |\hat{f}'(x+h)| - |\hat{f}'(x)| \right) \leq \frac{2\varepsilon}{h} \]

total computational error

\[ \frac{Mh}{2} + \frac{2\varepsilon}{h} \]

\[ f(h) = \frac{M}{2} h + 2\varepsilon h^{-1} \Rightarrow f'(h) = M - 2\varepsilon h^{-2} = 0 \Rightarrow \frac{M}{2} = 2\varepsilon h^2 \]

\[ h^2 = \frac{4\varepsilon}{M} \]
- all numerical values, input, intermediate, and output are rounded

Example: truncation vs. rounding error

- tradeoff between rounding error and truncation error when using finite-precision, floating-point arithmetic
- problem: computing the change in the surface area $A$ of the Earth if its radius $r \approx 6370$ km changes by a given amount $\Delta r$. Two different formulas are used:
  - one from geometry, $\Delta A = 4 \pi (r + \Delta r)^2 - 4 \pi r^2$, that is theoretically exact (assuming perfect real arithmetic), and
  - for small $\Delta r$, large rounding error -> inaccurate
  - for large $\Delta r$, small rounding error -> accurate
  - the other a simple approximation derived from calculus, $\Delta A \approx 8 \pi r \Delta r$, whose accuracy depends on the amount by which the radius changes
    - for small $\Delta r$, small truncation error -> accurate
    - for large $\Delta r$, large truncation error -> inaccurate