Homework 3
CS 210

Question	Points	Score
1	10	
2	15	
3	10	
4	5	
5	10	
6	5	
7	5	
8	5	
9	5	
10	10	
Total	80	

Singular Value Decomposition

1. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

$$\text{(a)} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{(e)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

2. Let A be an $m \times n$ singular matrix of rank r with SVD

$$A = U\Sigma V^T = \left(\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{array} \right) \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first r columns of U, $\tilde{U}$ consists of the remaining $m - r$ columns of U, $\hat{V}$ consists of the first r columns of V, and $\tilde{V}$ consists of the remaining $n - r$ columns of V. Give bases for the spaces range(A), null(A), range(A^T) and null(A^T) in terms of the components of the SVD of A, and a brief justification.

3. Use the SVD of A to show that for an $m \times n$ matrix of full column rank n, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A).

Least Squares

4. Consider the least squares problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2$. Which of the following statements are necessarily true?

   (a) If $\mathbf{x}$ is a solution to the least squares problem, then $A\mathbf{x} = \mathbf{b}$.

   (b) If $\mathbf{x}$ is a solution to the least squares problem, then the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ is in the nullspace of A^T.

   (c) The solution is unique.

   (d) A solution may not exist.

   (e) None of the above.

5. (Heath 3.3) Set up the linear least squares system $A\mathbf{x} \approx \mathbf{b}$ for fitting the model function $f(t, \mathbf{x}) = x_1 t + x_2 e^t$ to the three data points $(1, 2), (2, 3), (3, 5)$. Is the least squares solution unique? Why or why not?

6. (Heath 3.5) Let $\mathbf{x}$ be the solution to the linear least squares problem $A\mathbf{x} \approx \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

   Let $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ be the corresponding residual vector. Which of the following three vectors is a possible value for $\mathbf{r}$? Why?

   (a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

Orthogonal and Householder Matrices

7. (Heath 3.23) Which of the following matrices are orthogonal?

   (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$

   (d) $\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

8. (Heath 3.24) Which of the following properties does an orthogonal $n \times n$ matrix necessarily have? (Circle all that apply.)

   (a) It is nonsingular.

   (b) It preserves the Euclidean vector norm when multiplied times a vector.

   (c) Its transpose is its inverse.

   (d) Its columns are orthonormal.

   (e) It is symmetric.

   (f) It is diagonal.

   (g) Its Euclidean matrix norm is 1.

   (h) Its Euclidean condition number is 1.

9. A Householder matrix H

   (a) has condition number 1.

   (b) has the property $||H||_2 = 1$.

   (c) is uniquely defined by $H\mathbf{x} = \mathbf{b}$ for two vector $\mathbf{x}$ and $\mathbf{b}$ such that $||\mathbf{x}||_2 = ||\mathbf{b}||_2$.

   (d) Both (a) and (b).

   (e) All of the above.

10. Show that a $n \times n$ Householder matrix $H = I - 2\mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v}$ has an eigenvalue of 1 with multiplicity $n-1$ and an eigenvalue of -1 with multiplicity 1.