$\boxed{\text{Conditioning of Mat-Vec Mult} \; Ax}$

Given x, compute Ax:

perturb input $x + \Delta x$

$$\max_{\Delta x} \frac{\| A(\cancel{x} + \Delta x) - \cancel{A x} \| / \| Ax \|}{\| \cancel{x} + \Delta x - \cancel{x} \| / \| x \|} =$$

$$= \max_{\Delta x} \frac{\| A \Delta x \|}{\| \Delta x \|} \cdot \frac{\| x \|}{\| Ax \|}$$

$$\leq \| A \| \cdot \frac{\| x \|}{\| Ax \|} = (*)$$

$$\| A^{-1} \| = \max_{y \neq 0} \frac{\| A^{-1} y \|}{\| y \|}$$

let $Ax = y$, $A^{-1} y = x$

$$\| A^{-1} \| = \max_{x \neq 0} \frac{\| x \|}{\| Ax \|}$$

$$= \left(\min_{x \neq 0} \frac{\| Ax \|}{\| x \|} \right)^{-1}$$

and using $\dfrac{\| x \|}{\| Ax \|} \leq \| A^{-1} \|$

$$(*) \leq \| A \| \cdot \| A^{-1} \|$$

$\boxed{\text{Conditioning of } Ax = b}$

given b, find x s.t. $Ax = b$. $\Leftrightarrow$
equivalent to find $x = A^{-1} b$ (for A nonsingular)

$\therefore$ cond $\leq \| A^{-1} \| \cdot \| A \|$

$$\triangleq \quad \text{cond}(A) = \| A \| \cdot \| A^{-1} \|$$

A singular $\overset{\text{convention}}{\Longrightarrow}$ $\text{cond}(A) = \infty$

$\boxed{\text{Intuition re cond}(A)}$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$ ← Maximum stretch

$$\|A^{-1}\| = \max_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|} = \max_{x \neq 0} \frac{\|x\|}{\|Ax\|}, \text{ where } Ax = y$$

$$= \left(\min_{x \neq 0} \underbrace{\frac{\|Ax\|}{\|x\|}}_{\text{Minimum stretch}} \right)^{-1}$$

So

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\| = \max_{x \neq 0}$$

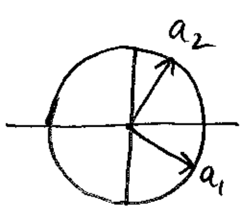
$$= \frac{\max_{x \neq 0} \|Ax\|/\|x\|}{\min_{x \neq 0} \|Ax\|/\|x\|} = \frac{\text{maximum stretch}}{\text{minimum stretch}}$$

How much ~~A~~ mult by A <u>distorts unit sphere</u>

A maps unit sphere to a hyperellipse

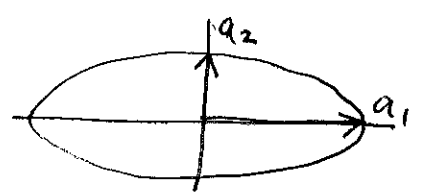
<u>Example 2.6</u>

$$A = \begin{bmatrix} 0.87 & 0.5 \\ -.5 & .87 \end{bmatrix}$$



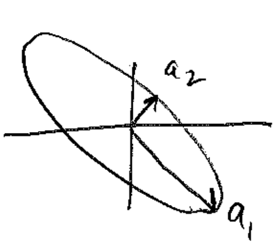
cond $(A) = 1$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix}$$



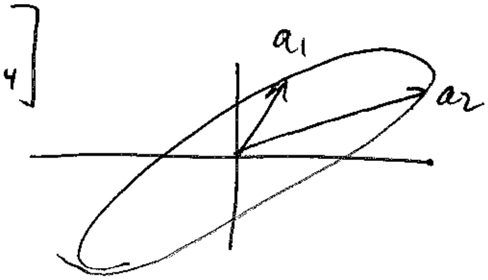
cond$(A) = 4$

$$A = \begin{bmatrix} 1.73 & 0.25 \\ -1 & 0.43 \end{bmatrix}$$



cond$(A)=4$

$$A = \begin{bmatrix} 1.52 & 0.91 \\ 0.47 & 0.94 \end{bmatrix}$$



cond$(A)=4$

$\boxed{\text{Ex. 2.5}}$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} .5 & 1.5 & -.5 \\ -.5 & 2.5 & -.5 \\ -.5 & -.5 & .5 \end{bmatrix}$$

$\|A\|_1 = 6$ $\qquad$ $\|A^{-1}\|_1 = 4.5$

$\|A\|_\infty = 8$ $\qquad$ $\|A^{-1}\|_\infty = 3.5$

$$\mathrm{cond}_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1 = 6 \cdot 4.5 = 27$$

$$\mathrm{cond}_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 8 \cdot 3.5 = 28$$

Note: - generally not practical to compute cond(A) from def'n like this example

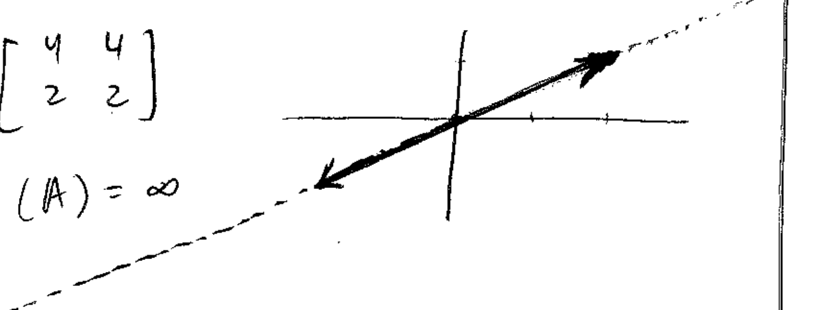
- usually get good estimate of cond(A) as by product of solution process.

$\boxed{\text{Intuition re cond(A)}}$

How close to singular?

$A = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$

cond(A) = ∞



but det(A) not indicative of cond(A)

E.g.

$\mathrm{cond}(\alpha I) = 1$

but

$\det(\alpha I) = \alpha^n$

Properties

1. $\text{cond}(A) \geq 1$
2. $\text{cond}(I) = 1$
3. $\text{cond}(\alpha A) = \text{cond}(A)$
4. D diag $\text{cond}(D) = \dfrac{\max_i |d_i|}{\min_i |d_i|}$

§2.3.4. Error Bounds

$\hat{x} = x + \Delta x$ $\boxed{\text{perturb } b}$ what can we say about $\dfrac{\|\Delta x\|}{\|x\|}$?

$$A\hat{x} = A(x+\Delta x) = Ax + A\Delta x = b + \Delta b$$

$$\Rightarrow A\Delta x = \Delta b \Rightarrow \Delta x = A^{-1}\Delta b$$

$$\|b\| = \|Ax\| \leq \|A\|\,\|x\| \Rightarrow \|x\| \geq \|b\| / \|A\|$$

$$\|\Delta x\| = \|A^{-1}\Delta b\| \leq \|A^{-1}\|\,\|\Delta b\| \Rightarrow \|\Delta x\| \leq \|A^{-1}\|\,\|\Delta b\|$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\,\|\Delta b\|}{\|b\|/\|A\|} = K(A)\frac{\|\Delta b\|}{\|b\|}$$

$$\boxed{\frac{\|\Delta x\|}{\|x\|} \leq K \frac{\|\Delta b\|}{\|b\|}}$$

K is amplification factor

$\boxed{\text{perturb } A}$

$$(A + \Delta A)(x + \Delta x) = b$$

$$\cancel{Ax} + (\Delta A)x + \Delta A\Delta x + A\Delta x = \cancel{b}.$$

$$\Delta A(x + \Delta x) + A\Delta x = 0$$

$$A^{-1}\Delta A(x + \Delta x) + \Delta x = 0.$$

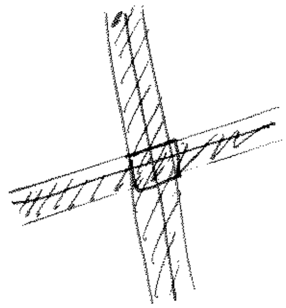
$$\|\Delta x\| \leq \|A^{-1}\|\,\|\Delta A\|\,\|\hat{x}\|$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \|A^{-1}\|\,\|\Delta A\|$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \leq K(A)\frac{\|\Delta A\|}{\|A\|}$$

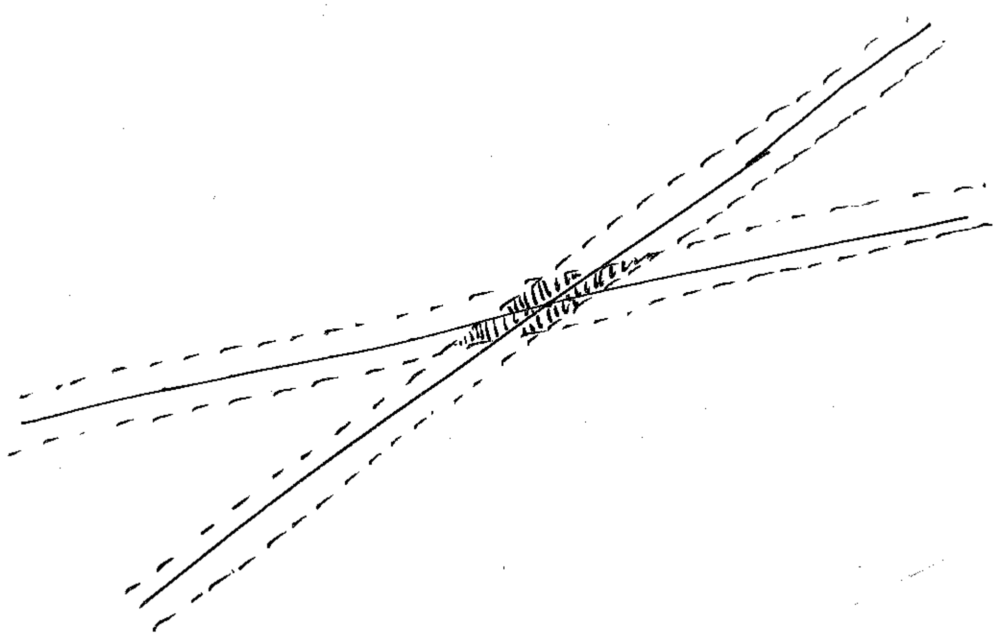
perturb b and A

$$\frac{\|\Delta x\|}{\|x\|} \le K(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|} \right)$$



well-conditioned

ill-conditioned



Summary: if input data accurate to machine precision, then

$$\frac{\| \hat{x} - x \|}{\| x \|} \lessapprox \text{cond}(A)\, \varepsilon_{mach}$$

computed solution loses

$$\log_{10}(\text{cond}(A))$$

digits of accuracy

Residual

$\hat{x}$: approximate solution to $Ax = b$

$$r = b - A\hat{x}$$

For A nonsingular, $\|r\| = 0 \iff \|\Delta x\| = \|\hat{x} - x\| = 0$

But small residual alone does not imply small error

$$\begin{aligned} r &= b - A\hat{x} \\ &= Ax - A\hat{x} \\ &= A(x - \hat{x}) \\ &= -A\,\Delta x \end{aligned}$$

$$\Delta x = -A^{-1} r$$

$$\|\Delta x\| \le \|A^{-1}\| \cdot \|r\|$$

$$\Rightarrow \quad \frac{\|\Delta x\|}{\|\hat{x}\|} \le \frac{\|A^{-1}\| \cdot \|A\| \cdot \|r\|}{\|A\| \cdot \|\hat{x}\|}$$

relative residual

Note: Scaling equations changes residual norm but not error.

$\therefore$ use relative res.

$$\frac{\|r\|}{\|A\| \cdot \|\hat{x}\|}$$

$$\boxed{\frac{\|\Delta x\|}{\|\hat{x}\|} \le \text{cond}(A) \frac{\|r\|}{\|A\| \cdot \|\hat{x}\|}}$$

small relative residual & well-conditioned A

$\Rightarrow$ small relative error

What if relative residual is large?

Let E be s.t.

$$(A+E)\hat{x} = b$$

$$\|r\| = \|b - A\hat{x}\| = \|E\hat{x}\| \le \|E\| \cdot \|\hat{x}\|$$

$$\Rightarrow \quad \frac{\|r\|}{\|A\| \cdot \|\hat{x}\|} \le \frac{\|E\|}{\|A\|}$$

large relative residual $\Rightarrow$ large backward error

Ex. 2.8 Small Residual

$$Ax = \begin{bmatrix} .913 & .659 \\ .457 & .330 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .254 \\ .127 \end{bmatrix} = b$$

$$\hat{x}_1 = \begin{bmatrix} -0.0827 \\ 0.5 \end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}$$

$$\|r_1\|_1 = 2.1 \cdot 10^{-4} \qquad \|r_2\|_1 = 2.4 \cdot 10^{-2}$$

exact $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

surprisingly $\hat{x}_2$ is better solution

(A is ill-conditioned).