By multiplying columns of $U\Sigma$ by rows of V^T

$$A = (U\Sigma)V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

Example

$$AV = U\Sigma \qquad \underbrace{\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}}_{A} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{V} = \underbrace{\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 3\sqrt{5} & \\ & \sqrt{5} \end{pmatrix}}_{\Sigma}$$

$$\sigma_1 = 3\sqrt{5}, \quad \sigma_2 = \sqrt{5}$$

$$A = U\Sigma V^T$$

$$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3\sqrt{5} & \\ & \sqrt{5} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} = 3\sqrt{5}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \sqrt{5}\frac{1}{\sqrt{10}}\frac{1}{\sqrt{2}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$= \frac{3}{2}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$$

$$|A^TA - \lambda I| = \begin{vmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{vmatrix} = (25-\lambda)^2 - 20^2 = 0$$

$$= \lambda^2 - 50\lambda + 25^2 - 20^2 = 0$$

$$= \lambda^2 - 50\lambda + 5^2(5^2 - 4^2) = 0$$

$$= \lambda^2 - 50\lambda + 5^2 3^2 = 0$$

$$\lambda_{1,2} = \frac{50 \pm \sqrt{50^2 - 4 \cdot 5^2 \cdot 3^2}}{2} = 25 \pm \sqrt{25^2 - 5^2 3^2}$$

$$= 25 \pm 5\sqrt{5^2 - 3^2} = 25 \pm 5\sqrt{4^2}$$

$$= 45, 5$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{45} = 3\sqrt{5}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{5}$$

Some SVD's

(1) $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = 3 e_1 e_1^T + 2(-e_2)(e_2^T)$

$\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A = U \Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \checkmark$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$A = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V^T} = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \checkmark$

$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V^T} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_2 = \sqrt{1+1} = \sqrt{2}$

$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{V^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \checkmark$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

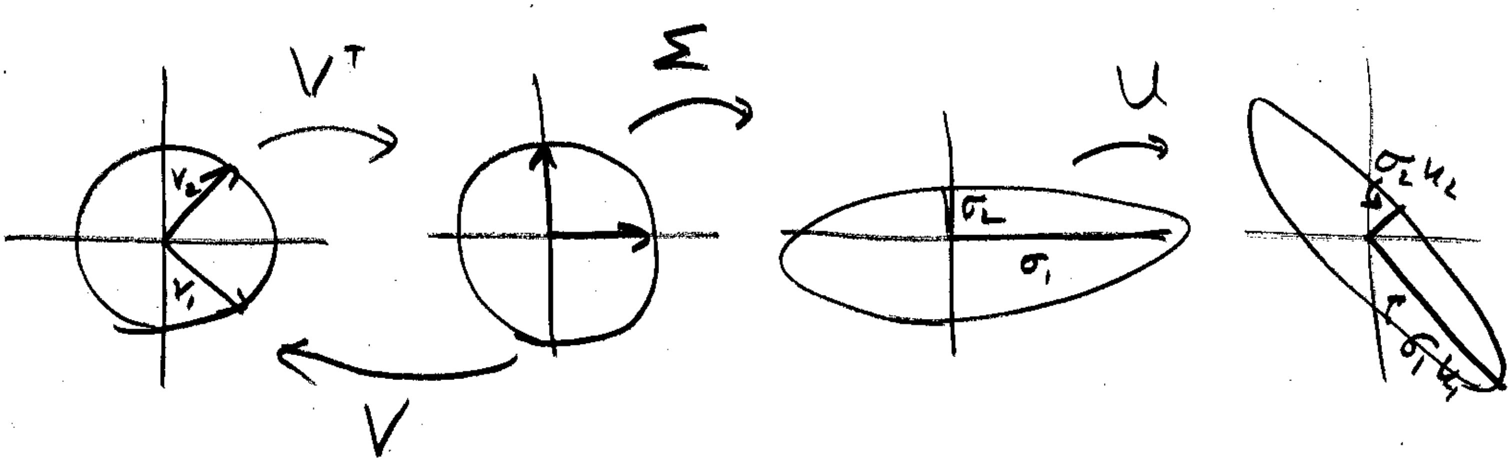
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Q. If $S = Q\Lambda Q^T$, what is its SVD?
is s.p.d.

Q. If $S = Q\Lambda Q^T$ is indefinite, what is its SVD?

Q. If $A = xy^T$, what is its SVD?

Geometry of SVD



SVD and rank

$$\underset{m\times n}{A} = \underset{m\times m}{U}\;\underset{m\times n}{\Sigma}\;\underset{n\times n}{V^T}$$

$m > n$

The SVD is a rank-revealing factorization.

The rank of A is the number of positive singular values.

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} = \left[\begin{array}{cccc|ccc} | & | & & | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n & \vec{u}_{n+1} & \cdots & \vec{u}_m \\ | & | & & | & | & & | \end{array}\right] \left[\begin{array}{cccc} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ \hline & & 0 & \end{array}\right] \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix}$$

A is full rank if $\operatorname{rank}(A) = n$

(or $\operatorname{rank}(A) = \min(m,n)$ more generally)

Then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

i.e., all the singular values are positive.

$$\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \\ | & & | & | & & | \end{bmatrix} \left[\begin{array}{cccccc} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline & & & 0 & & \end{array}\right] \begin{bmatrix} - & \vec{v}_1^T & - \\ & \vdots & \\ - & \vec{v}_r^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix}$$

A is rank deficient if $\operatorname{rank}(A) < n$

(or $\operatorname{rank}(A) < \min(m,n)$ more generally)

Then $\underbrace{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r}_{\text{positive singular values}} > \underbrace{\sigma_{r+1} = \cdots = \sigma_n}_{\text{zero singular values}} = 0$

In particular, $\operatorname{rank}(A) = r$ when there are exactly r positive singular values

Let A be an $m \times n$ matrix with rank r. Then the SVD of A looks like

$$\underset{m\times n}{A} = \underset{m\times m}{\left(\begin{array}{cccc|ccc} | & | & & | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \\ | & | & & | & | & & | \end{array}\right)} \underset{m\times n}{\left(\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_r & & & \\ \hline & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right)} \underset{n\times n}{\left(\begin{array}{c} - \; v_1^T \; - \\ - \; v_2^T \; - \\ \vdots \\ - \; v_r^T \; - \\ \hline - \; v_{r+1}^T \; - \\ \vdots \\ - \; v_n^T \; - \end{array}\right)}$$

- Note that for the full SVD, Σ will have the same shape as A $(m \times n)$, and U is square $m \times m$, V is square $n \times n$.
- Above, I have partitioned the columns of U into the sets $\{\vec{u}_1, \dots, \vec{u}_r\}$ and $\{\vec{u}_{r+1}, \dots, \vec{u}_m\}$, and the rows of V^T into the sets $\{\vec{v}_1^T, \dots, \vec{v}_r^T\}$, and $\{\vec{v}_{r+1}^T, \dots, \vec{v}_n^T\}$.
- Recall that we have 4 fundamental subspaces associated with A:
  - range (or columnspace) of A
  - nullspace A
  - range A^T
  - nullspace A^T

Fundamental Subspaces of A and the SVD $A = U\Sigma V^T$

- We can associate the 4 fundamental subspaces of A with the sets $\{u_1, \dots, u_r\}$, $\{u_{r+1}, \dots, u_m\}$, $\{v_1, \dots, v_r\}$, and $\{v_{r+1}, \dots, v_n\}$.

① columnspace of A.

These are all the vectors $A\vec{x}$ for any $\vec{x}$.

$$A\vec{x} = \sum_{i=1}^{r} \sigma_i u_i v_i^T x = \sum_{i=1}^{r} \underbrace{\sigma_i (v_i^T x)}_{\text{scalar}} \underbrace{\vec{u}_i}_{\text{vector}}$$

So a vector in colspace(A) is a linear combination of $\boxed{\{\vec{u}_1, \dots, \vec{u}_r\}}$

② nullspace A

These are all $\vec{z}$ s.t. $A\vec{z} = \vec{0}$

$$A\vec{z} = \sum_{i=1}^{r} \sigma_i (v_i^T z) \vec{u}_i + \sum_{i=r+1}^{n} 0 \cdot (v_i^T z) u_i + \sum_{i=n+1}^{m} 0 \, \vec{u}_i$$

For $A\vec{z} = \vec{0}$, we need $v_i^T z = 0$, $i = 1, \dots, r$

That is $\vec{z} \perp \vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$. We have a basis for this space for $\vec{z}$. It is $\boxed{\{\vec{v}_{r+1}, \dots, \vec{v}_n\}}$

So far, we have found

$$\underset{m\times n}{A} = \left(\begin{array}{cccc|ccc} \vert & \vert & & \vert & \vert & & \vert \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \\ \vert & \vert & & \vert & \vert & & \vert \end{array}\right) \left(\begin{array}{c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & 0 \\ \hline 0 & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} - \vec{v}_1^T - \\ - \vec{v}_2^T - \\ \vdots \\ - \vec{v}_r^T - \\ \hline - \vec{v}_{r+1}^T - \\ \vdots \\ - \vec{v}_n^T - \end{array}\right)$$

basis for range of A

basis for nullspace(A)

By looking at A^T

$$\underset{n\times m}{A^T} = \left(\begin{array}{ccc|ccc} \vert & & \vert & \vert & & \vert \\ \vec{v}_1 & \cdots & \vec{v}_r & \vec{v}_{r+1} & \cdots & \vec{v}_n \\ \vert & & \vert & \vert & & \vert \end{array}\right) \left(\begin{array}{c|c|c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} & 0 \end{array}\right) \left(\begin{array}{c} - \vec{u}_1^T - \\ - \vec{u}_2^T - \\ \vdots \\ - \vec{u}_r^T - \\ \hline - \vec{u}_{r+1}^T - \\ \vdots \\ - \vec{u}_m^T - \end{array}\right)$$

range(A^T)

nullspace (A^T)

we can identify in the same way

(3) columnspace of A^T

basis : $\{\vec{v}_1, \ldots, \vec{v}_r\}$

(4) nullspace A^T

basis : $\{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$

Since U and V are orthogonal matrices, they have orthonormal columns.

Therefore $\{\vec{u}_1, \dots, \vec{u}_r\} \perp \{\vec{u}_{r+1}, \dots, \vec{u}_m\}$

and $\{\vec{v}_1, \dots, \vec{v}_r\} \perp \{\vec{v}_{r+1}, \dots, \vec{v}_n\}$

So we can see that

- $\text{range}(A) \perp \text{nullspace}(A^T)$,

and

- $\text{range}(A^T) \perp \text{nullspace}(A)$
  (rowspace(A))

This is summarized in the picture from Strang: