CS 210
Practice Final

Name	
Student ID	
Signature	

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

True/False

For each True/False question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) Given a real, symmetric, invertible matrix A, power iteration, inverse power iteration, and Rayleigh quotient iteration are all algorithms that can be used to find an eigenvalue of A.

2. (T/F) If $A \in \mathbb{R}^{n \times n}$ is symmetric, then its singular value decomposition is the same as its eigenvalue decomposition.

3. (T/F) The Steepest Descent Method for unconstrained minimization breaks the problem down into a sequence of 1D problems.

4. (T/F) In the IEEE 754 floating point standard, denormalized floating point numbers are necessarily smaller in magnitude than the underflow level.

5. (T/F) The condition number of an invertible matrix A is the same as the condition number of A^{-1}.

6. (T/F) Examples of unstable algorithms include Gaussian Elimination without pivoting and classical Gram-Schmidt orthogonalization.

7. (T/F) The normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ for the least squares problem $A\mathbf{x} \approx \mathbf{b}$ may not have any solution at all.

Multiple Choice

For each Multiple Choice question, circle exactly one of (a) - (e).

8. Consider a matrix $A \in \mathbb{R}^{n \times n}$. Which of the following statements regarding its eigenvalues and eigenvectors true?

   I. An eigenvector corresponding to a given eigenvalue is unique.

   II. Scaling a matrix by a constant c will scale its eigenvalues by that constant.

   III. If a matrix has an eigenvalue of 0, then it is not invertible.

   (a) I only
   (b) II only
   (c) III only
   (d) II and III only
   (e) I, II and III

9. Which one statement about floating point numbers is true?

   (a) Floating point addition is commutative and associative.
   (b) Floating point numbers are distributed uniformly througout their range.
   (c) Subtracting one floating point number from another cannot cause overflow.
   (d) While double precision floating point offers more digits of precision than single precision floating point, they both have the same underflow and overflow levels.
   (e) None of the above.

10. Which of the following statements are true?

    I. Evaluating $\tan(x)$ near the vertical asymptote at $x = \pi/2$ is ill-conditioned.
    II. An unstable algorithm may compute a solution that has a large backward error.
    III. LU factorization without pivoting is unstable.

    (a) I only
    (b) II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

11. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. For each of the following statements regarding the Conjugate Gradient Method (CG), indicate whether the statement is true or false.

    (T/F) Minimizing the quadratic function $\mathbf{f}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathbf{c} = 0$ is equivalent to solving $A\mathbf{x} = \mathbf{b}$.
    (T/F) CG uses the local residual as the search direction in each iteration.
    (T/F) For solving the linear system $A\mathbf{x} = \mathbf{b}$, CG theoretically converges in at most n iterations, though in practice it may require more.
    (T/F) CG requires the full history of search directions in order to $A-$orthogonalize the new search direction in a Gram-Schmidt type approach.
    (T/F) Two search directions $\mathbf{s}_i$ and $\mathbf{s}_j$ generated by CG will necessarily satisfy $\mathbf{s}_i^T\mathbf{s}_j = 0$ in exact arithmetic.

12. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2$ are true?

    I. A solution of the LS problem satisfies $A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0}$.
    II. If A is rank-deficient, the least squares problem has no solution.
    III. $\|\mathbf{b} - A\mathbf{x}\|_2 = \|M(\mathbf{b} - A\mathbf{x})\|_2$, where M is an elimination matrix as used in LU factorization.

    (a) I only
    (b) II only
    (c) III only
    (d) I and II only
    (e) II and III only

Written Response

13. Consider a normalized floating point number system with p digits of precision, base β and integer exponent E, $L \leq E \leq U$.

    (a) How many normalized numbers are representable by the system?

    (b) What is the underflow level?

    (c) What is the overflow level?

    (d) How many additional numbers can be represented by allowing denormalized numbers?

14. *Least squares.* Let $A \in \mathbb{R}^{m \times n}$, where $m > n$. Consider the least squares (LS) problem

$$\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2.$$

(a) Assume A has full rank. Show how you would use the QR decomposition $A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$ to solve the LS problem.

(b) Now assume A is rank-deficient with rank $r < n$. Show how you would use the Singular Value Decomposition $A = U\Sigma V^T$, with $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, to solve the LS problem.

(c) In parts (a) and (b) is the solution unique? Why or why not?

(d) What does it say about $\mathbf{b}$ if $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2 = 0$?