CS 210
Midterm

Fall 2019

Name	
Student ID	
Signature	

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

Question	Points	Score
1.	2	
2.	2	
3.	2	
4.	2	
5.	2	
6.	2	
7.	4	
8.	4	
9.	4	
10.	4	
11.	4	
12.	16	
13.	12	
Total	60	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. ([✓ T]/F) Examples of unstable algorithms include Gaussian Elimination without pivoting and classical Gram-Schmidt orthogonalization.
2. (T/[✓ F]) If A is singular, then $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.
3. (T/[✓ F]) Solving $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n\times n}$ a diagonal matrix, requires $\sim n^2$ operations.
4. (T/[✓ F]) Cholesky factorization of a symmetric, positive definite matrix requires pivoting to be stable.
5. (T/[✓ F]) An invertible square matrix A does not have any 0 singular values, but can have both positive and negative singular values.
6. ([✓ T]/F) A Householder matrix is an orthogonal matrix with a determinant of -1.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

7. Which one of the following statements is false?

   (a) $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of a real matrix A.

   (b) [✓ $\|A^{-1}\|_2 = \frac{1}{\sigma_1}$, where σ_1 is the largest singular value of an invertible, real matrix A.]

   (c) If $\|\cdot\|_q$ and $\|\cdot\|_p$ are both vector p-norms, then they are equivalent, i.e., there exist constants C_1 and C_2 such that $C_1\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq C_2\|\mathbf{x}\|_q$ for all vectors $\mathbf{x}$.

   (d) An orthogonal matrix, Q, satisfies $\|Q\|_2 = 1$.

   (e) For any vector $\mathbf{x}$, $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$.

8. Let A be an $n \times n$ matrix. Which of the following properties would necessarily imply that A is singular?

   I. The rows of A are linearly dependent.

   II. A zero diagonal element is encountered while performing LU factorization (without pivoting).

   III. $A\mathbf{z} = \mathbf{0}$, for some $\mathbf{z} \neq \mathbf{0}$.

   (a) II only

   (b) I and II only

   (c) [✓ I and III only]

   (d) II and III only

   (e) I, II and III

9. Which of the following statements is false?

   (a) The number of solutions of $A\mathbf{x} = \mathbf{b}$ may depend on $\mathbf{b}$.

   (b) If A is singular, then $A\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.

   (c) If $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$, then $\mathbf{b}$ must be in the columnspace of A.

   (d) Solving a triangular system by forward or backward substitution requires $O(n^2)$ flops.

   (e) [✓ The LU and Cholesky factorizations of a symmetric, positive definite matrix would be exactly the same.]

10. Which of the following statements about the Singular Value Decomposition (SVD) are true?

    I. Every real matrix has an SVD.

    II. If a matrix Q is orthogonal, then its singular values are all 1.

    III. A matrix with rank r will have exactly r singular values that are greater than 0.

    (a) I only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) $\boxed{\text{I, II and III}}$

11. Let $A = U\Sigma V^T$ be the Singular Value Decomposition (SVD) of the matrix $A \in \mathcal{R}^{m\times n}$ and let A^+ denote the pseudoinverse of A. Which of the following statements are true?

    I. $\text{rank}(A) \leq \min(m, n)$.

    II. $A^+ = V\Sigma^+ U^T$ where Σ^+ is the pseudoinverse of Σ.

    III. The columns of U are mutually orthogonal, but need not be of unit length.

    (a) I only
    (b) II only
    (c) III only
    (d) $\boxed{\text{I and II only}}$
    (e) I and III only

Written Response

12. *Linear Systems and LU Factorization.*

   (a) Given a nonsingular system of linear equations $A\mathbf{x} = \mathbf{b}$, what effect on the solution vector $\mathbf{x}$ results from each of the following actions?

      i. Permuting the rows of $[A \quad \mathbf{b}]$.

      ii. Permuting the columns of A.

      iii. Multiplying both sides of the equation from the left by a nonsingular matrix M.

   (b) Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 11 & 15 \\ 6 & 15 & 23 \end{pmatrix}.$$

      i. Find unit lower triangular matrix L and upper triangular matrix U such that $A = LU$.

      ii. Use the factorization $A = LU$ that you found above to express A as $A = LDL^T$, where D is a diagonal matrix.

      iii. Use the factorization $A = LDL^T$ to find the Cholesky factorization of A, i.e., $A = R^TR$, where R is an upper triangular matrix.

   (c) Explain how you would use the factors L and U to solve the linear equations $A\mathbf{x} = \mathbf{b}$.

Solution:

(a) i. No effect on the solution vector $\mathbf{x}$.

   ii. If the columns of A are permuted as AP, then the rows of $\mathbf{x}$ should be permuted as $P^T\mathbf{x}$.

   iii. No effect on the solution vector $\mathbf{x}$.

(b) i.

$$\begin{aligned} A &= \begin{pmatrix} 2 & 4 & 6 \\ 4 & 11 & 15 \\ 6 & 15 & 23 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}}_{U} \end{aligned}$$

   ii. To find $A = LDL^T$, we will try to write $U = DL^T$ by pulling out a row scaling matrix D

from U to leave something unit upper triangular:

$$\begin{aligned} A &= \begin{pmatrix} 2 & 4 & 6 \\ 4 & 11 & 15 \\ 6 & 15 & 23 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}}_{U} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T} \end{aligned}$$

iii. To get the Cholesky factorization of A from the factorization $A = LDL^T$, we do the following:

$$\begin{aligned} A &= LDL^T \\ &= LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T \\ &= (LD^{\frac{1}{2}})(D^{\frac{1}{2}}L^T) \\ &= R^TR. \end{aligned}$$

For the matrix above, this process gives

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 2\sqrt{2} & \sqrt{3} & 0 \\ 3\sqrt{2} & \sqrt{3} & \sqrt{2} \end{pmatrix}}_{R^T} \underbrace{\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{2} \end{pmatrix}}_{R} \end{aligned}$$

(c) Using $A = LU$, rewrite $A\mathbf{x} = \mathbf{b}$ as

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ LU\mathbf{x} &= \mathbf{b} \\ L\mathbf{y} &= \mathbf{b}, \text{where } \mathbf{y} = U\mathbf{x}. \end{aligned}$$

Now lower triangular solve

$$L\mathbf{y} = \mathbf{b}$$

for $\mathbf{y}$ by forward substitution. Then upper triangular solve

$$U\mathbf{x} = \mathbf{y}$$

for $\mathbf{x}$ by backward substitution.

13. *SVD* Let A be an $n \times n$ matrix. A right inverse of A is a matrix B such that

$$AB = I,$$

and a left inverse of A is a matrix C such that

$$CA = I.$$

When A is full rank, then it has both right and left inverses and they are equal, i.e., $B = C = A^{-1}$. However, numerically, the left inverse is not necessarily a good right inverse and vice versa, as we will now demonstrate.

Let $A = U\Sigma V^T$, where U and V are $n \times n$ orthogonal matrices

$$U = \begin{pmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \\ | & | & & | \end{pmatrix}, \quad V = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix},$$

and Σ is an $n \times n$ diagonal matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

(a) Give an explicit expression for A^{-1}.

(b) Let $X = A^{-1} + \epsilon \mathbf{v}_n \mathbf{u}_1^T$, where $\epsilon \in \mathbb{R}$. Compute AX and XA. Express your answer as a rank-1 perturbation of the identity (i.e., in the form $I + \alpha \mathbf{u}\mathbf{v}^T$ for some scalar α, and unit vectors $\mathbf{u}$, and $\mathbf{v}$).

(c) Given any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, show that $||\mathbf{u}\mathbf{v}^T||_2 = ||\mathbf{u}||_2 ||\mathbf{v}||_2$. (Hint: recall that the 2-norm of a matrix is given by a its largest singular value).

(d) Use the above result to compute $||AX - I||_2$ and $||XA - I||_2$. What does this say about the accuracy of X as a left and right inverse?

Solution:

(a) If $A = U\Sigma V^T$ is the SVD of A, then

$$A^{-1} = V\Sigma^{-1}U^T$$

(b) Find AX:

$$\begin{aligned} AX &= A(A^{-1} + \epsilon \mathbf{v}_n \mathbf{u}_1^T) \\ &= I + \epsilon A \mathbf{v}_n \mathbf{u}_1^T \\ &= I + \epsilon U \Sigma V^T \mathbf{v}_n \mathbf{u}_1^T \\ &= I + \epsilon U \Sigma \hat{e}_n \mathbf{u}_1^T, \text{where } \hat{e}_n = (0, \ldots, 0, 1)^T \\ &= I + \epsilon U \sigma_n \hat{e}_n \mathbf{u}_1^T \\ &= I + \epsilon \sigma_n \mathbf{u}_n \mathbf{u}_1^T \end{aligned}$$

Find XA:

$$\begin{aligned} XA &= (A^{-1} + \epsilon \mathbf{v}_n \mathbf{u}_1^T)A \\ &= I + \epsilon \mathbf{v}_n \mathbf{u}_1^T A \\ &= I + \epsilon \mathbf{v}_n \mathbf{u}_1^T U \Sigma V^T \\ &= I + \epsilon \mathbf{v}_n \hat{e}_1^T \Sigma V^T \\ &= I + \epsilon \mathbf{v}_n \sigma_1 \hat{e}_1^T V^T \\ &= I + \epsilon \sigma_1 \mathbf{v}_n \mathbf{v}_1^T \end{aligned}$$

(c) We write the matrix $\mathbf{u}\mathbf{v}^T$ in SVD form:

$$\begin{aligned} \mathbf{u}\mathbf{v}^T &= \|\mathbf{u}\|_2 \frac{\mathbf{u}}{\|\mathbf{u}\|_2} \|\mathbf{v}\|_2 \frac{\mathbf{v}^T}{\|\mathbf{v}\|_2} \\ &= \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \frac{\mathbf{u}}{\|\mathbf{u}\|_2} \frac{\mathbf{v}^T}{\|\mathbf{v}\|_2} \end{aligned}$$

This is of the form $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T$ with $\sigma_1 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$. Therefore,

$$\|\mathbf{u}\mathbf{v}^T\|_2 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$

(d) Note, if A was invertible, then A^{-1} exists and

$$\begin{aligned} \|AA^{-1} - I\|_2 = 0 \\ \|A^{-1}A - I\|_2 = 0 \end{aligned}$$

X is an approximate inverse so $\|AX - I\|_2$ and $\|XA - I\|_2$ won't necessarily be zero but they should be small for an accurate approximation.
X as a right inverse:

$$\begin{aligned} \|AX - I\|_2 &= \|I + \epsilon \sigma_n \mathbf{u}_n \mathbf{u}_1^T - I\|_2 \\ &= \|\epsilon \sigma_n \mathbf{u}_n \mathbf{u}_1^T\| \\ &= \epsilon \sigma_n \end{aligned}$$

X as a left inverse:

$$\begin{aligned} \|XA - I\|_2 &= \|I + \epsilon \sigma_1 \mathbf{v}_n \mathbf{v}_1^T - I\|_2 \\ &= \|\epsilon \sigma_1 \mathbf{v}_n \mathbf{v}_1^T\|_2 \\ &= \epsilon \sigma_1 \end{aligned}$$

Therefore, the accuracy in using X as a right inverse is better than (or equal to) the accuracy in using X as a left inverse.