Lecture 1

- matrix-vector multiplication $A\vec{x}$
  - by rows
  - by columns
- column space of A
  - for one vector
  - for two linearly independent vectors
  - column spaces of $A \in \mathbb{R}^{3\times m}$
  - column spaces of invertible $A \in \mathbb{R}^{n\times n}$
- rank of A
  - dimension of column space of A
  - dimension of row space of A
- matrix-matrix multiplication AB
  - using inner products (row of A dot column of B)
  - using outer products (column of A outer proudct with row of B)
- CR factorization
- rank 1 matrix (outer product matrix)
  - as building blocks of matrices (e.g., CR, LU, eigenvalue decomposition, SVD, etc.)

Lecture 2

- four fundamental subspaces
  - range(A) (or columnspace(A))
  - nullspace(A)
  - range(A^T) (or rowspace(A))
  - nullspace(A^T)
- rowspace(A) $\perp$ nullspace(A)
- rowspace(A^T) $\perp$ nullspace(A^T)
- find columnspace or rowspace of a small matrix
- find nullspace of small or simple matrix
- rank(AB)
- rank(A+B)
- rank(A^TA) = rank(AA^T) = rank(A) = rank(A^T)

Lecture 3

- solving linear systems
  - algebraic interpretation: intersection of several lines (or hyperplanes) ("row view")
  - geometric interpretation: generate rhs vector $\vec{b}$ from columns of A ("column view")
  - A singular and how that relates to algebraic and geometric interpretations
- A nonsingular, relation to
  - determinant
  - inverse
  - rank
  - nullspace
- existence and uniqueness of solutions to $A\vec{x} = \vec{b}$.
- triangular system
  - upper triangular and forward substitution
  - lower triangular and backward substitution
- preconditioning $MA\vec{x} = M\vec{b}$
- permutation matrix P
  - row permutation PA
  - column permutation AP
- LU factorization
  - by hand for simple, small matrix
- LU (a.k.a. Gaussian Elimination) unstable without pivoting
- $PA = LU$ (LU with row pivoting)
- How to solve $A\vec{x} = \vec{b}$ with LU
- LU operation count
- triangular solve (forward and backward substitution) operation counts
- estimating operation counts

Lecture 4

- orthogonal vectors
- orthogonal basis for a subspace
- orthonormal basis
  - find components of vector $\vec{v}$ in an orthonormal basis
- orthogonal subspaces
- orthogonal matrix, Q
  - satisfies $Q^{-1} = Q^T$, i.e., $Q^TQ = QQ^T = I$
  - rotation or reflection
  - orthogonal 2×2 matrices
- vector 2-norm
  - triangle inequality
  - law of cosines
- projector matrix P
  - idempotence $P^2 = P$
- orthogonal vs. non-orthogonal projector
- complementary projector $I - P$
- Householder reflection matrix

Lecture 5

- $A\vec{v} = \lambda\vec{v}$
  - eigenvalue λ
  - eigenvector $\vec{v}$
- scaling eigenvectors ($\alpha\vec{v}$ is also an eigenvector)
- transformations
  - powers of A
  - inverse of A
  - similarity
  - scalar shift $(A + sI)$
- characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$

- finding eigenvalues and eigenvectors of a 2×2 matrix
  - quadratic formula
- geometric multiplicity
- algebraic multiplicity
- trace, determinant of A and eigenvalues
- eigenvalues of triangular matrix
- diagonalizing a matrix and eigenvalue decomposition
- real, symmetric matrix

Lecture 6

- spd matrices
- spd conditions
  - eigenvalues $\lambda_i > 0$
  - energy $\vec{x}^T S \vec{x} > 0 \quad \forall \vec{x}$
  - all leading determinants
  - pivots in elimination
- eigen decomposition of symmetric matrix
- expressing any vector in terms of the eigenvectors of A
- LDL^T factorization
- Cholesky factorization
  - operation count compared to LU
  - stability compared to LU

Lecture 7

- singular value decomposition (SVD): $A = U \Sigma V^T$
- singular vectors, $\vec{u}_i, \vec{v}_i$
- singular values, σ_i
- $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$
- reduced SVD: $A = U_r \Sigma_r V_r^T$

Lecture 8

- computing SVD for 2×2 matrices
- computing SVD for matrices that are almost in SVD form
- singular values of orthogonal matrix
- geometry of the SVD (maps unit circle to ellipse in 2D)
- SVD and rank
- four fundamental subspaces of A from the SVD
  - null space of A
  - row space of A (i.e., column space of A^T)
  - column space of A
  - null space of A^T
- orthogonal subspaces
  - null space $A \perp$ column space A^T
  - null space $A^T \perp$ column space A

Lecture 9

- vector norms
- vector norm properties
  - positive for non-zero vector
  - scale comes out
  - triangle inequality
- $\ell^1, \ell^2, \ell^\infty$ norms
- set of unit length vectors under $\ell^1, \ell^2, \ell^\infty$ norms
- p-norms
- 2-norm properties
  - $\vec{v} \cdot \vec{w} = \|\vec{v}\|_2 \|\vec{w}\|_2 \cos\theta$
  - $\vec{v} \cdot \vec{v} = \|\vec{v}\|_2^2$
- S-norm for symmetric matrix S
- matrix norms
- matrix norm properties

- positive for non-zero matrix
- scale comes out
- triangle inequality
- submultiplicative (for p-norms and Frobenius norm)

- induced matrix norms (1-, 2-, and ∞-norms)
  - 1-norm : max abs column sum
  - ∞-norm : max abs row sum
  - 2-norm : σ_1
- Frobenius norm
  - $\|A\|_F^2 = \text{trace}(A^T A)$
  - $\|A\|_F^2 = \sigma_1^2 + \ldots + \sigma_n^2$
- 2-norm and Frobenius norm invariant under orthogonal transformations

Lecture 10

- project one vector onto a direction given by a unit vector
- project one vector onto a direction given by an arbitrary vector
- Gram-Schmidt orthogonalization
- Modified Gram-Schmidt orthogonalization
- QR factorization
  - by Gram-Schmidt
  - by Householder