Strang I.5 Orthogonal Matrices and Subspaces

1. Orthogonal vectors $\vec{x}$ and $\vec{y}$

$$\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

$$\vec{x} \perp \vec{y} \iff \vec{x}^T\vec{y} = 0$$

(or if $\vec{x},\vec{y}$ complex, $\bar{x}^T y = 0$; recall: $\cancel{x = a+ib}$ $\bar{x} = a - ib$)

2. Orthogonal basis for a subspace:

$$\vec{v}_i^T\vec{v}_j = 0$$

orthonormal basis:

$$\vec{v}_i^T\vec{v}_j = 0 \quad \text{and} \quad \|v_i\|^2 = 1$$

3. Orthogonal subspaces R and N

Every $\vec{x} \in R$ and $\vec{y} \in N$ satisfy $\vec{x}^T\vec{y} = 0$

Example, row space and nullspace

$$\begin{bmatrix} - & a_1^* & - \\ - & a_2^* & - \\ & \vdots & \\ - & a_n^* & - \end{bmatrix}\begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | \\ 0 \\ | \end{bmatrix}$$

$$a_i^* x = 0$$

4. Tall, thin matrices Q with orthonormal columns : $Q^TQ = I$

$$Q^TQ = \begin{bmatrix} — & q_1^T & — \\ — & q_2^T & — \\ & \vdots & \\ — & q_n^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = I_n$$

$\|Qx\|_2 = \|x\|_2$ because

$$\|Qx\|_2^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|_2^2$$

Q is tall + thin, i.e., $m > n$

$QQ^T \neq I_m$

5. Orthogonal matrices

square, with orthonormal columns

$Q^T = Q^{-1}$

$Q^TQ = I$ and $QQ^T = I$

columns are an orthonormal basis for $\mathbb{R}^n$
rows " " " " " "

Vector 2-norm

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\vec{x}^T\vec{x}}$$

$$\vec{x}^T\vec{x} = \|x\|_2^2$$

Norm properties:

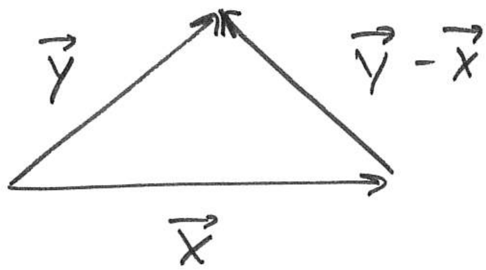
1. $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$

2. $\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$ subadditive or triangle ineq.

3. $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

1. Orthogonal vectors

<u>triangle</u>



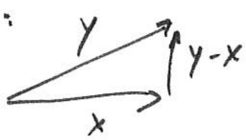
$$\|\vec{y} - \vec{x}\|_2^2 = (y-x)^T(y-x) = y^Ty - x^Ty - y^Tx + x^Tx$$

$$= \|y\|_2^2 + \|x\|_2^2 - 2x^Ty$$

Let $\|x\| = a$, $\|y\| = b$, $\|y-x\| = c$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

<u>right triangle</u> :



$c^2 = a^2 + b^2$ (Pythagorean theorem)

<u>other triangles</u> : (Law of cosines)

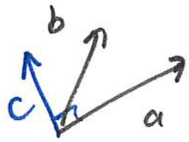
2. Orthogonal bases

standard basis, $\hat{\imath}, \hat{\jmath}, \hat{k}$, or $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Every subspace of $\mathbb{R}^n$ has an orthogonal basis

$\vec{a}, \vec{b}$



$$\vec{c} = b - \frac{a^Tb}{a^Ta}a$$

"Gram-Schmidt"

3. Orthogonal Subspaces

rowspace A $\perp$ nullspace A

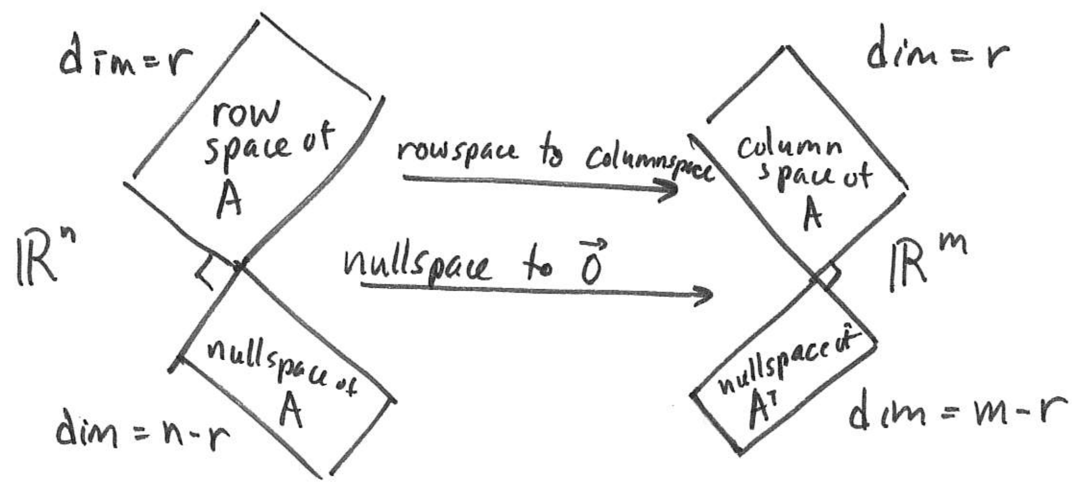
$$Ax = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} | \\ 0 \\ | \end{bmatrix}$$

columnspace A $\perp$ nullspace A^T

$$A^T y = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} \begin{bmatrix} | \\ y \\ | \end{bmatrix} = \begin{bmatrix} | \\ 0 \\ | \end{bmatrix}$$

The Singular Value Decomposition, SVD, finds orthonormal bases $\vec{v}_1, \dots, \vec{v}_r$ for the rowspace of A, and $\vec{u}_1, \dots, \vec{u}_r$ for the columnspace of A.

Also, $$A v_1 = \sigma_1 u_1, \quad A v_2 = \sigma_2 u_2, \quad \dots, \quad A v_r = \sigma_r u_r$$



4. Tall, thin Q w/ orthonormal columns : $Q^TQ = I$

Ex.

$$Q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad Q_2 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

$Q_1^T Q_1 = I_1$ $\qquad Q_2^T Q_2 = I_2$ $\qquad Q_3^T Q_3 = I_3$

$Q_1 Q_1^T \neq I_3$ $\qquad Q_2 Q_2^T \neq I_3$ $\qquad Q_3 Q_3^T = I_3$

Let $P = QQ^T$ $\qquad$ Projector

P is a "projection" $\Leftrightarrow$ P idempotent

$$P^2 = PP = Q\underbrace{Q^TQ}_{= I}Q^T = QQ^T = P \quad \checkmark$$

If furthermore $P = P^T$ (symmetric),

then P is an orthogonal projector

Proof: $\quad Px \perp (I-P)x \ \forall x \Longleftrightarrow P = P^T$

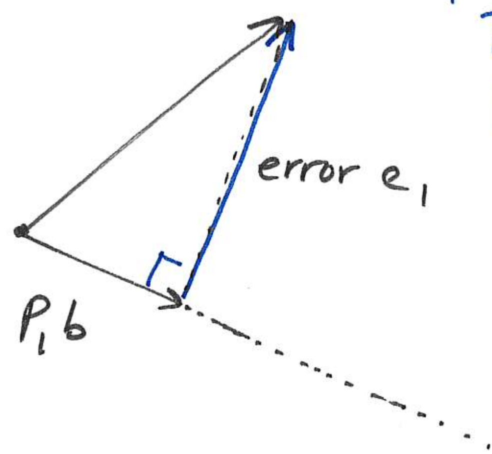
$\Rightarrow \quad Px \perp (I-P)x \Rightarrow x^TP^T(I-P)x = 0 \Rightarrow x^TP^Tx = x^TP^TPx$

$\Rightarrow P^T = P^TP \Rightarrow \langle \text{transpose} \rangle \ P = P^TP = P^T \checkmark.$

$\Leftarrow \quad P = P^T \Rightarrow x^TPx = x^TP^Tx \Rightarrow x^TPx = x^TP^TPx$

Example 1

$$\vec{b} = \begin{pmatrix}3\\3\\3\end{pmatrix}$$



projection of $\vec{b}$ onto line

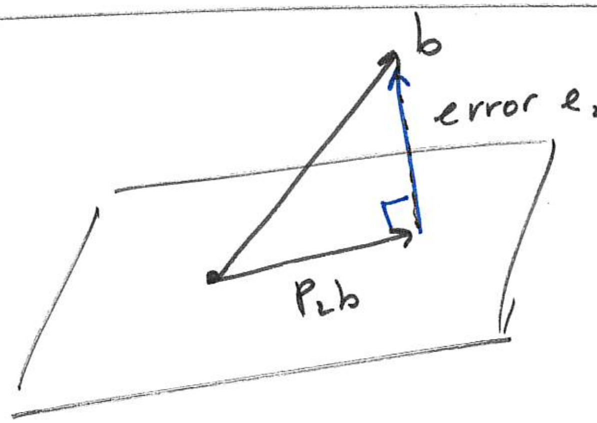
$$P_1\vec{b} = Q_1Q_1^T\vec{b}$$

$$Q_1Q_1^Tb = \frac{1}{\cancel{3}}\begin{pmatrix}2\\2\\-1\end{pmatrix}\frac{1}{\cancel{3}}\cancel{9} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$$

$$e_1 = (I - P_1)\vec{b} = (I - Q_1Q_1^T)b = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix}$$

$$\|e_1\|_2 = \sqrt{18}$$

$$P_2\vec{b} = Q_2Q_2^Tb$$



$$P_2b = \frac{1}{9}\begin{pmatrix}2&2\\2&-1\\-1&2\end{pmatrix}\begin{pmatrix}2&2&-1\\2&-1&2\end{pmatrix}\begin{pmatrix}3\\3\\3\end{pmatrix}$$

$$= \frac{1}{\cancel{9}}\begin{pmatrix}2&2\\2&-1\\-1&2\end{pmatrix}\begin{pmatrix}\cancel{9}1\\\cancel{9}1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}$$

projection of $\vec{b}$ onto plane

$$e_2 = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}4\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

$$\|e_2\|_2 = \sqrt{9}$$

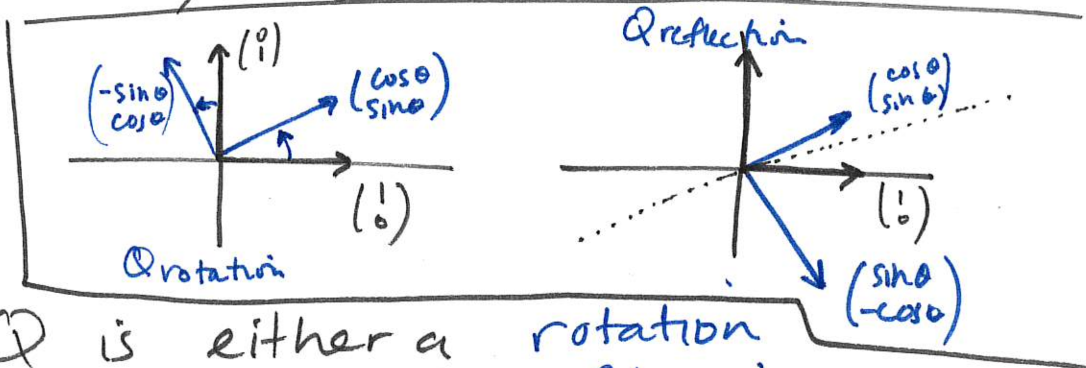
Q. What is P_3b? What is e_3?

5. Orthogonal Matrices

Q is square, $Q^{-1} = Q^T$

$Q^T Q = I$

$Q Q^T = I$



<u>In 2D</u> Q is either a rotation or a reflection

$$Q_{\text{rotation}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \text{CCW rotation by angle } \theta$$

$$Q_{\text{reflection}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \text{reflection across the } \frac{\theta}{2}\text{-line}$$

Q_1, Q_2 orthogonal $\Rightarrow$ $Q_1 Q_2$ orthogonal

<u>Proof:</u> $(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = I$

$$\left.\begin{array}{l} \text{reflection} \cdot \text{reflection} = \text{rotation} \\ \text{rotation} \cdot \text{rotation} = \text{rotation} \\ \text{rotation} \cdot \text{reflection} = \text{reflection} \end{array}\right\} \text{ still true in } \mathbb{R}^n$$

Orthogonal Basis = Orthogonal Axes in $\mathbb{R}^n$

$$Q = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix}, \quad Q \text{ orthogonal}$$

$\forall\, v \in \mathbb{R}^n$

$$\vec{v} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \ldots + \alpha_n \vec{q}_n \qquad (*)$$

How to find α_k?

$$\vec{q}_k^{\,T} \vec{v} = \alpha_1 \vec{q}_k^{\,T} \vec{q}_1 + \alpha_2 \vec{q}_k^{\,T} \vec{q}_2 + \ldots + \alpha_n \vec{q}_k^{\,T} \vec{q}_n$$

$$= \alpha_k$$

$$(*) \Rightarrow \quad \vec{v} = Q \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

So

$$Q^T \vec{v} = Q^T Q \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

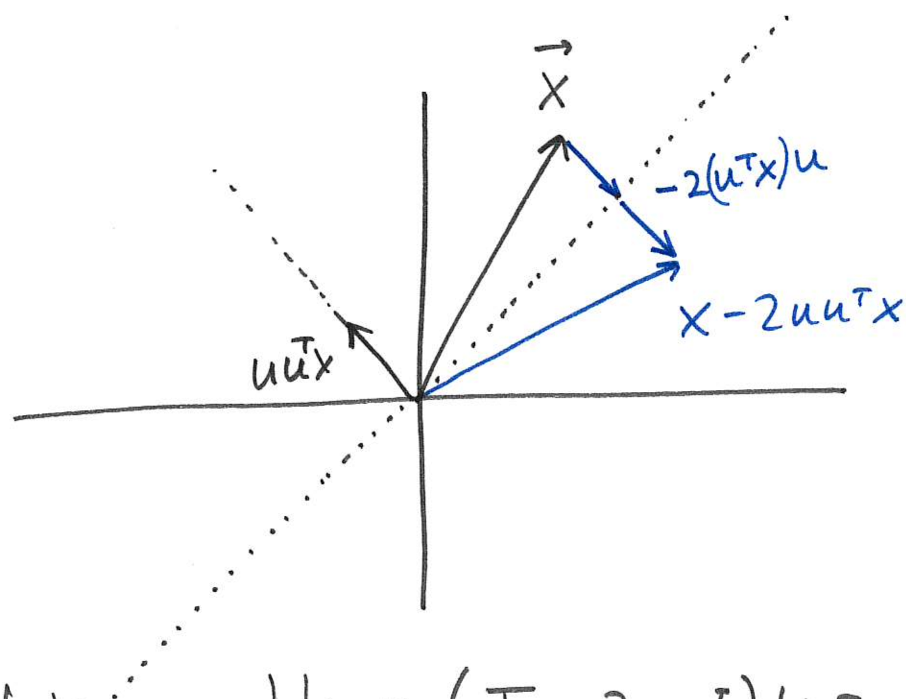
Householder Reflections

$\vec{u}$ unit vector

$$H = I - 2uu^T$$

H symmetric: $H^T = I - 2uu^T$ ✓

H orthogonal:
$$\begin{aligned} H^2 &= (I - 2uu^T)(I - 2uu^T) \\ &= I - 4uu^T + 4uu^Tuu^T \\ &= I - 4uu^T + 4uu^T = I \end{aligned}$$



<u>Note</u>:
- $Hu = (I - 2uu^T)u = u - 2uu^Tu$
  $= u - 2u = -u$
- for $v \perp u$, $Hv = (I - 2uu^T)v = v$