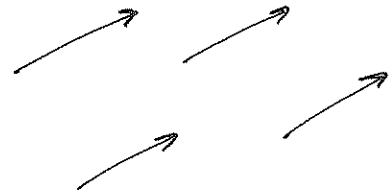
Lecture 3 Notes — Math Review

1. points — locations $\cdot P$ $\cdot Q$

   P, Q, R

2. vectors — direction & magnitude $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$
   - no notion of location
   - all relative



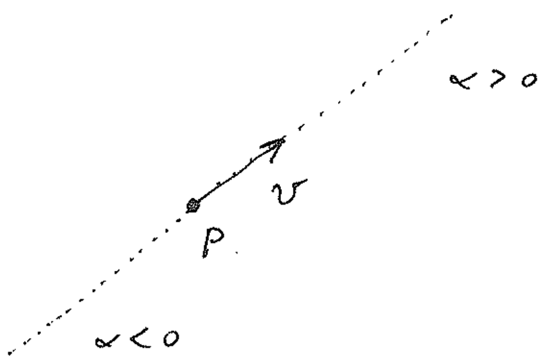
   - vector addition
   - $(+,-)$ scalar multiplication   scalars : α, β, γ
   - vector space
     - coordinate system & basis vectors.

$$\boxed{\underset{\sim}{a} = (a_1, a_2, a_3) = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}}$$

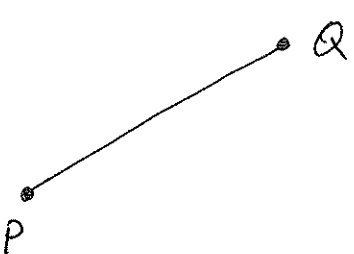
3. vector space vs. affine space
   └ point + vector

4. lines

$$P(\alpha) = P + \alpha \underset{\sim}{v}$$



line segments



$$(1-\alpha)P + \alpha Q$$

$$0 \leq \alpha \leq 1$$

equivalent :

$$P + \alpha\underbrace{(Q - P)}_{= \underset{\sim}{v}}$$

5. Dot Product

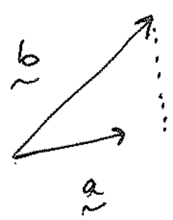
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

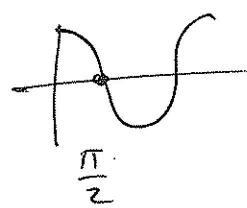
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} \qquad \underset{1\times 3}{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}$$

$$\mathbf{a} \cdot \mathbf{a} = \underbrace{a_1 a_1 + a_2 a_2 + a_3 a_3}_{a_1^2 + a_2^2 + a_3^2} = \|a\|^2$$

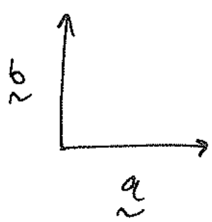
$$\mathbf{a} \cdot \mathbf{b} = \|a\| \, \|b\| \cos\theta$$

<u>geometric interpretation</u>: ($\mathbf{a}$ unit vector)



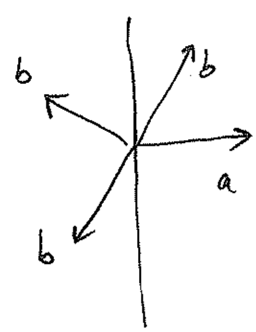


$\boxed{Q.}$

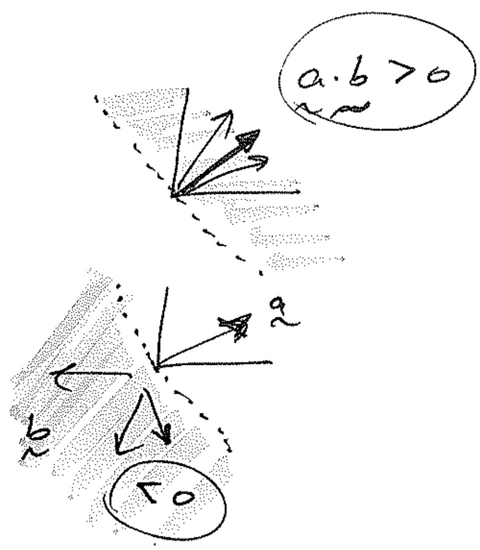


$$\mathbf{a} \cdot \mathbf{b} = ?$$

$\boxed{Q}$



$$\mathbf{a} \cdot \mathbf{b} \; \overset{?}{\substack{> \\ < \\ =}} \; 0$$



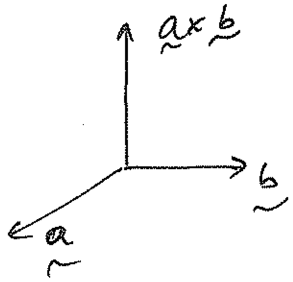
region of
−ive
dot product

Cross Product

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\,\hat{i} + (a_3 b_1 - a_1 b_3)\,\hat{j} + (a_1 b_2 - a_2 b_1)\,\hat{k}$$

result of cross product is another vector!

Right-hand rule:

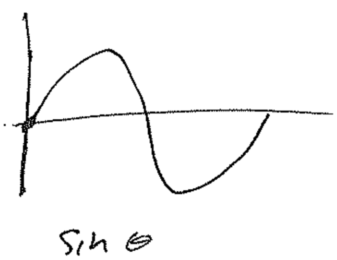


$$\|\underset{\sim}{a} \times \underset{\sim}{b}\| = \|a\|\,\|b\| \sin\theta$$

magnitude of the resulting vector

direction is given by right-hand rule.

$\boxed{Q}$ $\underset{\sim}{a} \times \underset{\sim}{a} = ?$



$= 0$

Points are locations

Use uppercase for points

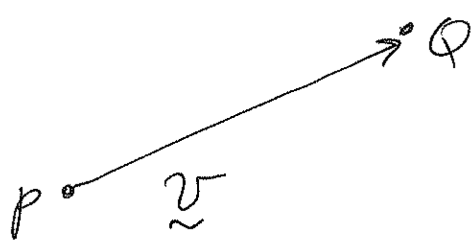
Vectors

- Cartesian "canonical" basis 3D.

$$\hat{i}, \hat{j}, \hat{k}$$

- other bases can be used

$$\underset{\sim}{v} = Q - P$$



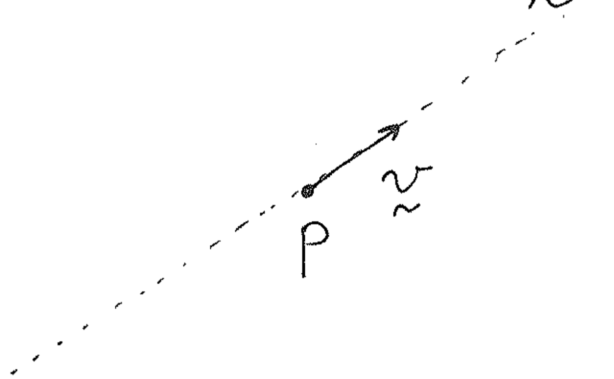
$$\underline{Q} = P + \underset{\sim}{v}$$

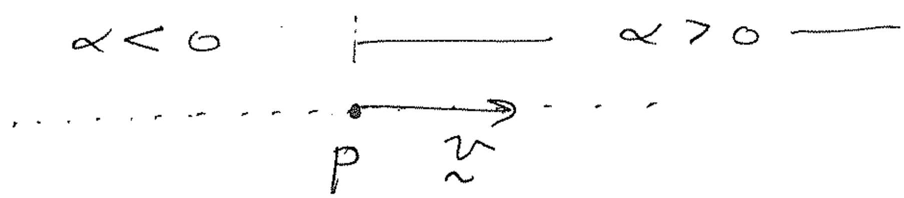
line is <u>1D</u>

because it can be parametrized by 1 number.

$\hookrightarrow \alpha$

$$p(\alpha) = P + \alpha \underset{\sim}{v}$$





matrix mult.
as a sequence
of transformations

$$f_1(x) = Ax$$

$$f_2(x) = Bx$$

$$f(x) = f_1(f_2(x)) = A(f_2(x))$$

$$= (AB)x$$

$$C = AB$$

$$f(x) = (AB)x$$

$$f(x) = Cx$$

$$f_1 \circ f_2$$

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transpose of a matrix

$$a_{ji} \leftarrow a_{ij}$$

matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

a_{ij}

ith row

j^{th} column

$$\text{2 rows} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

3 columns

2×3 matrix

matrix multiplication

$$\begin{array}{cc} A & B \\ m \times k & k \times n \end{array}$$

- you can't just multiply any two matrices
- they have to be $\boxed{\text{compatible}}$.

$$y = Ax$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} x_2 + \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} x_3$$

$f(x)$

linear if & only if

① $f(\alpha x) = \alpha f(x)$

② $f(x+y) = f(x) + f(y)$

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vector

point

$f(x) = mx + b$

AFFINE

① $f(\alpha x) = m\alpha x + b = \alpha\left(mx + \frac{b}{\alpha}\right)$

$\neq \alpha f(x)$

② $f(x+y) = m(x+y) + b$

$= mx + my + b$

$\neq f(x) + f(y)$

"linear in x"

"linear" (highest order term is linear).

$b = 0$

LINEAR

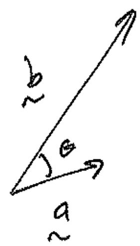
$f(x) = mx$

① $f(\alpha x) = m\alpha x = \alpha(mx)$

$= \alpha f(x)$ ✓.

② $f(x+y) = m(x+y) = mx + my$ ✓.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\| \, \|\underset{\sim}{b}\| \cos\theta$$



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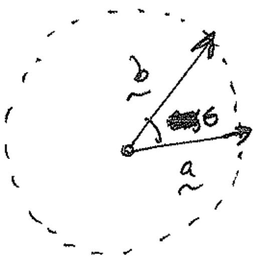
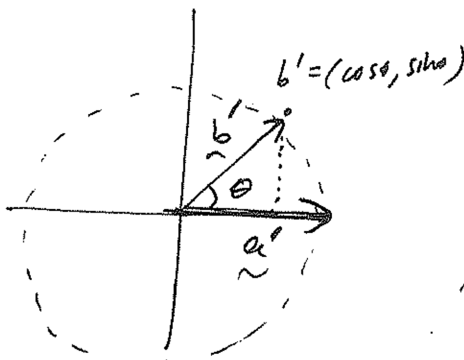
① assume $\underset{\sim}{a}$, $\underset{\sim}{b}$ are unit vectors



$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \cos\theta$$



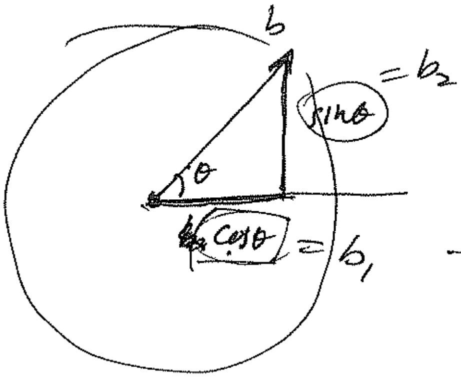
$$\underset{\sim}{a}' = (1, 0)$$

$$\underset{\sim}{b}' = (b_1', b_2')$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = b_1' = \cos\theta$$

rotate $\underset{\sim}{a}$ & $\underset{\sim}{b}$ so $\underset{\sim}{a}$ is on x-axis

→ haven't changed θ!



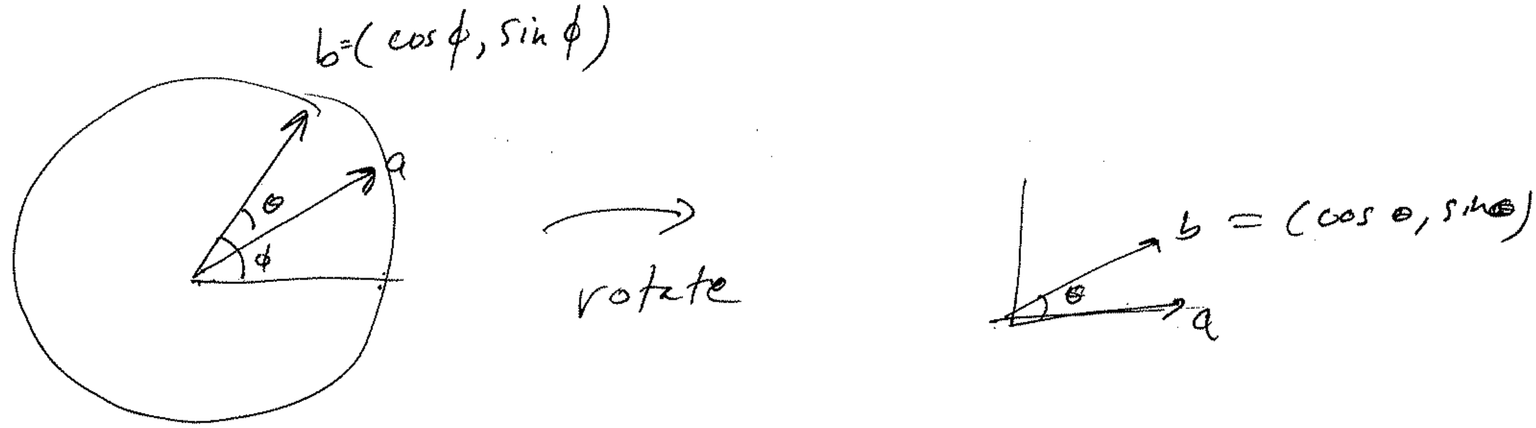
R rotation

$$\underset{\sim}{a}' = R\underset{\sim}{a}$$

$$\underset{\sim}{b}' = R\underset{\sim}{b}$$

$$(R\underset{\sim}{a}) \cdot (R\underset{\sim}{b}) = (R\underset{\sim}{a})^T (R\underset{\sim}{b})$$

$$= \underset{\sim}{a}^T R^T R \underset{\sim}{b} = \underset{\sim}{a}^T \underset{\sim}{b}$$



b=(cos ϕ, sin ϕ)
a
θ
ϕ
rotate
b = (cos θ, sin θ)
θ
a