general rotations
Rotation

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

X axis

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Y axis

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Z axis

The rows and columns are orthonormal
Rotation about an arbitrary axis

Rotating about an axis by theta degrees

- Rotate about x to bring axis to xz plane
- Rotate about y to align axis with z-axis
- Rotate theta degrees about z
- Unrotate about y, unrotate about x

\[ M = \text{Rx}^{-1} \text{Ry}^{-1} \text{Rz}(\theta) \text{Ry} \text{Rx} \]

- Can you determine the values of Rx and Ry?
Composite Transformations

• Rotating about a fixed point
  - **basic** rotation alone will rotate about origin
    but we want:
Composite Transformations

- Rotating about a fixed point
- Move fixed point \((px, py, pz)\) to origin
- Rotate by desired amount
- Move fixed point back to original position

\[
M = T(px, py, pz) \, R_z(\theta) \, T(-px, -py, -pz)
\]
euler angles
Euler Angles

- A general rotation is a combination of three elementary rotations: around the x-axis (x-roll), around the y-axis (y-pitch) and around the z-axis (z-yaw).
Gimbal and Euler Angles

Z-X'-Z'’
extrinsic - rotations about the reference axes
intrinsic - rotations about the object fixed axes
http://www.youtube.com/watch?v=zc8b2Jo7mno
quaternions
Here as he walked by on the 16th of October 1843 Sir William RowanHamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

\[ i^2 = j^2 = k^2 = ij = jk = ki = -1 \]

Cuts on a stone over a bridge.
Quaternions

- axis/angle representation
- interpolates smoothly
- easy to compose

<whiteboard>
Quaternion Interpolation

linear: treat quaternions as 4-vectors, note non-uniform speed
spherical linear: constant speed
Rotations in Reality

- It’s easiest to express rotations in Euler angles or Axis/angle

- We can convert to/from any of these representations

- Choose the best representation for the task
  - input: Euler angles
  - interpolation: quaternions
  - composing rotations: quaternions, orientation matrix