Triangles
barycentric coordinates
barycentric coordinates
barycentric coordinates

\[ \beta = -1 \quad \beta = 0 \quad \beta = 1 \]
barycentric coordinates

\[ \gamma = 2 \]

\[ \gamma = 1 \]

\[ \gamma = 0 \]

\[ \gamma = -1 \]
barycentric coordinates
barycentric coordinates

$$p = \alpha a + \beta b + \gamma c$$

What are $$(\alpha, \beta, \gamma)$$?

<whiteboard>
2.7 Triangles

Area:
\[ \text{area} = \frac{1}{2} \begin{vmatrix} b-a & c-a \\ x_b-x_a & x_c-x_a \\ y_b-y_a & y_c-y_a \end{vmatrix} \]

Barycentric coordinates:

Assign a color at each vertex and smoothly interpolate.

Non-orthogonal coordinate system:

\[
\begin{align*}
\rho &= a + \beta(b-a) + \gamma(c-a) \\
\rho &= a - \beta a + \beta b + \gamma c - \beta a \\
\rho &= a - \beta a - \gamma a + \beta b + \gamma c \\
\rho &= (1-\beta-\gamma)a + \beta b + \gamma c \\
\rho &= \alpha a + \beta b + \gamma c
\end{align*}
\]
Inside test

\[ p = \alpha a + \beta b + \gamma c \]

\[ p \text{ inside triangle} \]

\[ \alpha + \beta + \gamma = 1 \]

\[ 0 < \alpha, \beta, \gamma < 1 \]

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**Edges**

\[ \beta = 0 \]

\[ \gamma = 0 \]

\[ \alpha = 0 \]

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**Vertices**

\[ \delta = 1 \]

\[ \alpha = \beta = 0 \]

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Interpolation

use \( \alpha, \beta, \gamma \) to interpolate properties.
Find $\alpha$, $\beta$, $\gamma$ given a point $p$

![Diagram of triangle with points a, b, c, and p]

\[ p = a + \beta(b-a) + \gamma(c-a) \]

Solve for $\beta$, $\gamma$

\[
\begin{bmatrix}
  b-a \\
  c-a \\
  1
\end{bmatrix}
\begin{bmatrix}
  \beta \\
  \gamma \\
  1
\end{bmatrix} =
\begin{bmatrix}
  p-a \\
  p-1 \\
  1
\end{bmatrix}
\]

Or use geometric property: bary. coords. are signed, scale distance from corresponding edge.

\[ \beta = \frac{\text{fac}(p)}{\text{fac}(b)} \]
\[ \alpha = \frac{A_a}{A}, \quad \beta = \frac{A_b}{A}, \quad \gamma = \frac{A_c}{A} \]
Barycentric Coordinates
CS 130

1. Want to interpolate vertex data along a segment

- Define \( f(x) \) for all points \( x \) on the line
- Value at endpoints: \( f_A, f_B \).
- Interpolation should get the endpoints right: \( f(A) = f_A, f(B) = f_B \)
- \( f(P) = \alpha f(A) + (1 - \alpha) f(B) \).
- \( 0 \leq \alpha \leq 1 \).
- Symmetry: define \( \beta = 1 - \alpha \).
- \( f(P) = \alpha f(A) + \beta f(B) \), with \( \alpha + \beta = 1 \).
- \( \alpha = \frac{\text{len}(PB)}{\text{len}(AB)}, \beta = \frac{\text{len}(AP)}{\text{len}(AB)} \)

2. Extend this to a triangle

- Define \( f(x) \) for all points \( x \) on the triangle
- Value at vertices: \( f_A, f_B, f_C \).
- Interpolation should get the vertices right: \( f(A) = f_A, f(B) = f_B, f(C) = f_C \)
- \( f(P) = \alpha f(A) + \beta f(B) + \gamma f(C) \), with \( \alpha + \beta + \gamma = 1 \).
• Weights form isocontours:

- \( \alpha > 1 \)
- \( \alpha = 1 \)
- \( 0 < \alpha < 1 \)
- \( \alpha = 0 \)
- \( \alpha < 0 \)

• Note that \( \alpha < 0 \) or \( \alpha > 1 \) lies outside the triangle

• Compute using distance to edge:

- \( \alpha = \frac{\text{len}(PF)}{\text{len}(AE)} = \frac{\frac{1}{2}\text{len}(PF)\text{len}(BC)}{\frac{1}{2}\text{len}(AE)\text{len}(BC)} = \frac{\text{area}(PBC)}{\text{area}(ABC)} \)

- Similarly: \( \beta = \frac{\text{area}(APC)}{\text{area}(ABC)} \), \( \gamma = \frac{\text{area}(ABP)}{\text{area}(ABC)} \)
• Pattern of areas
• Since \( \text{area}(PBC) + \text{area}(APC) + \text{area}(ABP) = \text{area}(ABC) \), we have \( \alpha + \beta + \gamma = 1 \)
• Barycentric interpolation is okay for \( z \)-values
• Barycentric interpolation is okay for colors in orthographic case
• Barycentric interpolation does not work for colors in the projective case

3. Inside/outside tests

• \( \alpha < 0 \) or \( \alpha > 1 \) lies outside the triangle (Same for \( \beta < 0 \) or \( \beta > 1 \), \( \gamma < 0 \) or \( \gamma > 1 \))
• Inside the triangle if \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \) and \( 0 \leq \gamma \leq 1 \).
• Sufficient to check \( \alpha, \beta, \gamma \geq 0 \)
• For example if \( \alpha \geq 0 \) and \( \beta \geq 0 \) then \( \gamma = 1 - \alpha - \beta \leq 1 - \beta \leq 1 \).
• Since we need the weights to compute the depth values when doing \( z \)-buffering, we might as well also use them to determine inside/outside.