Lecture 3 Notes - Math Review

1. Points — locations \( P, Q, R \)

2. Vectors — direction & magnitude \( \mathbf{u}, \mathbf{v}, \mathbf{w} \)
   - no notion of location
   - all relative
   - vector addition
   - scalar multiplication
     
     \( \alpha, \beta, \gamma \)

3. Vector space
   - coordinate system — basis vectors:
     \[ \mathbf{a} = (a_1, a_2, a_3) = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \]

4. Lines
   \[ P(\alpha) = P + \alpha \mathbf{v} \]

   Line segments
   \[ (1-\alpha)P + \alpha Q \]
   \[ 0 \leq \alpha \leq 1 \]

   equivalent:
   \[ P + \alpha (Q - P) \]
5. Dot Product

\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3
\]

\[
\mathbf{a} \cdot \mathbf{a} = \mathbf{a}^\top \mathbf{a}
\]

\[
\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = ||\mathbf{a}||^2
\]

\[
\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \, ||\mathbf{b}|| \cos \theta
\]

**Geometric Interpretation:**

\[
(\mathbf{a} \text{ unit vector})
\]

\[
\mathbf{b}
\]

\[
\mathbf{a}
\]

\[
Q
\]

\[
\mathbf{a} \cdot \mathbf{b} = ?
\]

\[
\mathbf{a} \cdot \mathbf{b} > 0
\]

\[
\mathbf{a} \cdot \mathbf{b} < 0
\]

**Region of Negative Dot Product**

\[
Q
\]
Cross Product

\[ \sim \times \sim = \begin{vmatrix} \hat{k} & \hat{j} & \hat{i} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k} \]

\text{result of cross product is another vector!}

Right-hand rule:

\[ \| \sim \times \sim \| = \| \sim \| \| \sim \| \sin \theta \]

\[ \vec{a} \times \vec{b} \]

\[ \angle \text{magnitude of the resulting vector} \]

\[ \text{direction is given by right-hand rule.} \]

Question:

\[ \sim \times \sim = ? \]

\[ \sim \times \sim = 0 \]

\[ \sin \theta \]
matrices

\[ A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} \]

\[ a_{ij} \]

2 rows \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{pmatrix}

3 columns

2x3 matrix

matrix multiplication

\[ A \times B \]

\[ m \times n \]

\[ \text{you can't just multiply any two matrices} \]
\[ \text{they have to be compatible} \]

\[ y = A \times x \]

\[ \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

\[ = \begin{pmatrix}
  a_{11} \\
  a_{21} \\
  a_{31}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
+ \begin{pmatrix}
  a_{12} \\
  a_{22} \\
  a_{32}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
+ \begin{pmatrix}
  a_{13} \\
  a_{23} \\
  a_{33}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} \]
\[ f_1(x) = Ax \]
\[ f_2(x) = Bx \]

\[ f(x) = f_1(f_2(x)) = A(f_2(x)) = (A B)x \]

\[ f(x) = (A B) x \]

\[ f(x) = Cx \]

\[ f_1 \circ f_2 \]

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**transpose of a matrix**

\[ a_{ji} \leftarrow a_{ij} \]