Proof of the Triangular Inequality of Motif Distance

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Notations

- N_A = number of edge points in image A
- AB_m= the set containing maximal number of edge points that match between A and B.
- $|AB_m|$ = size of the set AB_m

- In section 3.5 of our paper, we define:

$$D_{motif}(A,B) = (A+B)/2 - (N_A - \mathit{MUE}(A,B))$$

Now we can rewrite it as:

$$D_{motif}(A,B) = (A+B)/2 - |AB_m|$$

Given any three images A,B and C,

we are going to prove:

$$D(A,B) + D(B,C) \geq D(A,C).$$

By the distance definition,

we can have :

$$[(N_A+N_B)/2 - |AB_m|] + [(N_B+N_C)/2 - |BC_m|] \geq [(N_A+N_C)/2 - |AC_m|]$$

That is:

$$N_B \geq |AB_m| + |BC_m| - |AC_m| \qquad (*)$$

Lemma 1: $|AB_m \cap BC_m| \leq |AC_m|$

The proof is trivial. Because:

$$AB_m \cap BC_m \subseteq A, \quad AB_m \cap BC_m \subseteq C$$

And AC_m contains maximal matched edge points between A and C, so:

$$AB_m \cap BC_m \subseteq AC_m$$

Then:

$$|AB_m \cap BC_m| \leq |AC_m|$$

Lemma 2: $|AB_m \cap BC_m| \geq |AB_m| + |BC_m| - N_B$

$$|AB_m \cap BC_m| = N_B - |\overline{AB_m \cap BC_m}|$$

$$= N_B - |\overline{AB_m} \cup \overline{BC_m}|$$

$$\geq N_B - |\overline{AB_m} + \overline{BC_m}|$$

$$\geq N_B - [\,(N_B - |AB_m|) + (N_B - |BC_m|)\,]$$

$$\geq |AB_m| + |BC_m| - N_B$$

Combine Lemma 1 and 2, we can obtain the (*) on page 3.

Proof done.