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Greedy Set-Cover Algorithms (1974-1979, Chvátal, Johnson, Lovász, Stein)

Neal E. Young, University of California, Riverside
www.cs.ucr.edu/∼neal
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INDEX TERMS: dominating set, greedy algorithm, hitting set, set cover, minimizing a linear function subject to a submodular constraint

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1 PROBLEM DEFINITION

Given a collection $\mathcal{S}$ of sets over a universe U, a *set cover* $C \subseteq \mathcal{S}$ is a subcollection of the sets whose union is U. The *set-cover problem* is, given $\mathcal{S}$, to find a minimum-cardinality set cover. In the *weighted set-cover problem*, for each set $s \in \mathcal{S}$ a weight $w_s \geq 0$ is also specified, and the goal is to find a set cover C of minimum total weight $\sum_{s \in C} w_s$.

Weighted set cover is a special case of *minimizing a linear function subject to a submodular constraint*, defined as follows. Given a collection $\mathcal{S}$ of objects, for each object s a non-negative weight w_s, and a non-decreasing submodular function $f : 2^{\mathcal{S}} \rightarrow \mathbb{R}$, the goal is to find a subcollection $C \subseteq \mathcal{S}$ such that $f(C) = f(\mathcal{S})$ minimizing $\sum_{s \in C} w_s$. (Taking $f(C) = |\cup_{s \in C} s|$ gives weighted set cover.)

2 KEY RESULTS

The *greedy algorithm* for weighted set cover builds a cover by repeatedly choosing a set s that minimize the weight w_s divided by number of elements in s not yet covered by chosen sets. It stops and returns the chosen sets when they form a cover:

greedy-set-cover($\mathcal{S}$, w)
1. Initialize $C \leftarrow \emptyset$. Define $f(C) \doteq |\cup_{s \in C} s|$.
2. Repeat until $f(C) = f(\mathcal{S})$:
3. Choose $s \in \mathcal{S}$ minimizing the price per element $w_s / [f(C \cup \{s\}) - f(C)]$.
4. Let $C \leftarrow C \cup \{s\}$.
5. Return C.

Let H_k denote $\sum_{i=1}^{k} 1/i \approx \ln k$, where k is the largest set size.

Theorem 1. *The greedy algorithm returns a set cover of weight at most H_k times the minimum weight of any cover.*

Proof. When the greedy algorithm chooses a set s, imagine that it charges the price per element for that iteration to each element newly covered by s. Then the total weight of the sets chosen by the algorithm equals the total amount charged, and each element is charged once.

Consider any set $s = \{x_k, x_{k-1}, \ldots, x_1\}$ in the optimal set cover C^*. Without loss of generality, suppose that the greedy algorithm covers the elements of s in the order given: $x_k, x_{k-1}, \ldots, x_1$. At the start of the iteration in which the algorithm covers element x_i of s, at least i elements of s remain uncovered. Thus, if the greedy algorithm were to choose s in that iteration, it would pay a cost per element of at most w_s/i. Thus, in this iteration, the greedy algorithm pays at most w_s/i per element covered. Thus, it charges element x_i at most w_s/i to be covered. Summing over i, the total amount charged to elements in s is at most $w_s H_k$. Summing over $s \in C^*$ and noting that every element is in some set in C^*, the total amount charged to elements overall is at most $\sum_{s \in C^*} w_s H_k = H_k \textsc{opt}$. □

The theorem was shown first for the unweighted case (each $w_s = 1$) by Johnson [6], Lovász [9], and Stein [14], then extended to the weighted case by Chvátal [2].

Since then a few refinements and improvements have been shown, including the following:

Theorem 2. *Let $\mathcal{S}$ be a set system over a universe with n elements and weights $w_s \leq 1$. The total weight of the cover C returned by the greedy algorithm is at most $[1+\ln(n/\textsc{opt})]\textsc{opt}+1$ (compare to [13]).*

Proof. Assume without loss of generality that the algorithm covers the elements in order $x_n, x_{n-1}, \ldots, x_1$. At the start of the iteration in which the algorithm covers x_i, there are at least i elements left to cover, and all of them could be covered using multiple sets of total cost $\textsc{opt}$. Thus, there is some set that covers not-yet-covered elements at a cost of at most $\textsc{opt}/i$ per element.

Recall the charging scheme from the previous proof. By the preceding observation, element x_i is charged at most $\textsc{opt}/i$. Thus, the total charge to elements $x_n, \ldots, x_i$ is at most $(H_n - H_{i-1})\textsc{opt}$. Using the assumption that each $w_s \leq 1$, the charge to each of the remaining elements is at most 1 per element. Thus, the total charge to all elements is at most $i - 1 + (H_n - H_{i-1})\textsc{opt}$. Taking $i = 1 + \lceil \textsc{opt} \rceil$, the total charge is at most $\lceil \textsc{opt} \rceil + (H_n - H_{\lceil \textsc{opt} \rceil})\textsc{opt} \leq 1 + \textsc{opt}(1 + \ln(n/\textsc{opt}))$. □

Each of the above proofs implicitly constructs a linear-programming primal-dual pair to show the approximation ratio. The same approximation ratios can be shown with respect to any fractional optimum (solution to the fractional set-cover linear program).

Other results. The greedy algorithm has been shown to have an approximation ratio of $\ln n - \ln \ln n + O(1)$ [12]. For the special case of set systems whose duals have finite Vapnik-Chervonenkis (VC) dimension, other algorithms have substantially better approximation ratio [1]. Constant-factor approximation algorithms are known for geometric variants of the closely related k-median and facility location problems (see the K-median and Facility Location entry of this text).

The greedy algorithm generalizes naturally to many problems. For example, for minimizing a linear function subject to a submodular constraint (defined above), the natural extension of the greedy algorithm gives an H_k-approximate solution, where $k = \max_{s \in \mathcal{S}} f(\{s\}) - f(\emptyset)$, assuming f is integer-valued [10].

The set-cover problem generalizes to allow each element x to require an arbitrary number r_x of sets containing it to be in the cover. This generalization admits a polynomial-time $O(\log n)$-approximation algorithm [8].

The special case when each element belongs to at most r sets has a simple r-approximation algorithm [15, §15.2]. When the sets have uniform weights ($w_s = 1$), the algorithm reduces to the following: select any maximal collection of elements, no two of which are contained in the same set; return all sets that contain a selected element.

The variant "Max k-coverage" asks for a set collection of total weight at most k covering as many of the elements as possible. This variant has a $(1 - 1/e)$-approximation algorithm [15, Problem 2.18] (see [7] for sets with non-uniform weights).

For a general discussion of greedy methods for approximate combinatorial optimization, see [5, Ch. 4].

Finally, under likely complexity-theoretic assumptions, the $\ln n$ approximation ratio is essentially the best possible for any polynomial-time algorithm [3, 4].

3 APPLICATIONS

Set Cover and its generalizations and variants are fundamental problems with numerous applications. Examples include:

- selecting a small number of nodes in a network to store a file so that all nodes have a nearby copy,
- selecting a small number of sentences to be uttered to tune all features in a speech-recognition model [11],
- selecting a small number of telescope snapshots to be taken to capture light from all galaxies in the night sky,
- finding a short string having each string in a given set as a contiguous sub-string.

4 OPEN PROBLEMS [optional]

None to report.

5 EXPERIMENTAL RESULTS

None to report.

6 CROSS REFERENCES

EDITOR PLEASE FORMAT

Greedy Algorithms Entry 00423
Set Covering Entry 00275
K-median and Facility Location Entry 00479

7 RECOMMENDED READING

[1] Hervé Brönnimann and Michael T. Goodrich, Almost optimal set covers in finite VC-dimension. *Discrete & Computational Geometry*, 14(4):463–479, 1995.

[2] Vasek Chvátal. A greedy heuristic for the set-covering problem. *Mathematics of Operations Research*, 4(3):233–235, 1979.

[3] Carsten Lund and Mihalis Yannakakis. On the hardness of approximating minimization problems. *Journal of the ACM*, 41(5):960–981, 1994.

[4] Uriel Feige. A threshold of ln n for approximating set cover. *Journal of the ACM*, 45(4):634–652, 1998.

[5] Teo F. Gonzalez, *Handbook of Approximation Algorithms and Metaheuristics.* (Chapman & Hall/CRC Computer & Information Science Series), 2007.

[6] David S. Johnson. Approximation algorithms for combinatorial problems. *Journal of Computer and System Sciences*, 9:256–278, 1974.

[7] S. Khuller, A. Moss, and J. Naor. The budgeted maximum coverage problem. *Information Processing Letters*, 70(1):39-45, 1999.

[8] Stavros G. Kolliopoulos and Neal E. Young. Tight approximation results for general covering integer programs. In *Proceedings of the forty-second annual IEEE Symposium on Foundations of Computer Science*, 522–528, 2001.

[9] László Lovász. On the ratio of optimal integral and fractional covers. *Discrete Mathematics*, 13:383–390, 1975.

[10] George L. Nemhauser and Laurence A. Wolsey. *Integer and Combinatorial Optimization.* John Wiley & Sons, New York, 1988.

[11] J.P.H. van Santen and A.L. Buchsbaum. Methods for optimal text selection. In *Proceedings of the European Conference on Speech Communication and Technology (Rhodos, Greece)*, 2:553–556, 1997.

[12] Petr Slavik. A tight analysis of the greedy algorithm for set cover. *Journal of Algorithms*, 25(2):237–254, 1997.

[13] Aravind Srinivasan. Improved approximations of packing and covering problems. In *Proceedings of the twenty-seventh annual ACM Symposium on Theory of Computing*, 268–276, 1995.

[14] S. K. Stein. Two combinatorial covering theorems. *J. Comb. Theory A*, 16:391–397, 1974.

[15] V.V. Vazirani. *Approximation Algorithms*, Springer. 2001.