Deriving greedy algorithms and Lagrangian-relaxation algorithms

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February 16, 2007
set cover

standard randomized rounding
   existence proof
   method of conditional probabilities
   algorithm

iterated sampling
   existence proof
   method of conditional probabilities
   algorithm

vertex cover (duality)
   existence proof
   method of conditional probabilities
   algorithm
   implicit primal-dual algorithm

multicommodity flow
   existence proof
   algorithm for integer solution
   algorithm for fractional solution

lower bound on iterations

fast algorithm for explicitly given problems

two open questions
set cover
input: collection \( s_1, s_2, \ldots, s_m \) of sets over universe \( U \)

minimize \( \sum_{i=1}^{m} x_i \) \hspace{1mm} subject to
\[
(\forall e \in U) \sum_{s_i \ni e} x_i \geq 1
\]
\[
(\forall i) \hspace{1mm} x_i \in \{0, 1\}
\]

- Value of optimal fractional solution \( x^* \)
  is a lower bound on optimal integer solution.
a fractional set cover $x^*$

sets

elements

c,eb,d,ea,c,da,b,c
.3.7.7.3
11.41.311

d,ac,d,eb,c,da,b,c
.3.7.7.3
11.41.311

e,b,c,da,c,eb,d
.3.7.7.3
11.41.311

1 1 1.3 1.4 1
standard randomized rounding

Let $x^*$ be an optimal fractional set cover.

Let $\lambda = \ln 2n$.

For each set $s_i \in S$ independently do:

choose $s_i$ with probability $p_i \doteq \min\{\lambda x_i^*, 1\}$.

**Theorem**

*With positive probability, chosen sets form a cover of size at most $2 \ln(2n) \sum_i x_i^*$.***
Probability element $e$ not covered:

$$\prod_{s_i \ni e} 1 - p_i < \prod_{s_i \ni e} \exp(-\lambda x_i^*)$$

$$= \exp \left( -\lambda \sum_{s_i \ni e} x_i^* \right)$$

$$\leq \exp(-\lambda)$$

$$= 1/2n$$

Pr[ exists uncovered element ] $< 1/2$
Let $x^*$ be an optimal fractional set cover.
Let $\lambda = \ln 2n$.
For each set $s_i \in S$ independently do:
choose $s_i$ with probability $p_i \equiv \min\{\lambda x^*_i, 1\}$.

Expected number of sets chosen is

$$\sum_i p_i \leq \ln(2n) \sum_i x^*_i.$$ 

Pr[ more than $2 \ln(2n) \sum_i x^*_i$ sets chosen ] \leq 1/2
Theorem

With positive probability, chosen sets form a cover of size at most 2 \ln(2n) \sum_i x_i^*.

Proof.

Pr[exists uncovered element] < 1/2
Pr[more than 2 \ln(2n) \sum_i x_i^* sets chosen] \leq 1/2

Pr[chosen sets form cover of size \leq 2 \ln(2n) \sum_i x_i^*] > 0
method of conditional probabilities
converts existence proof into an efficient algorithm

Random experiment is random walk starting here.

Each node labeled with expected number of bad events if random walk was to start there.

Leaf label is number of bad events in that outcome.

Method of conditional probabilities chooses next node so labels don't increase...

... thus guarantees an outcome with no bad events.

Let $x^*$ be an optimal fractional set cover.
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For each set $s_j \in S$ independently do:
choose $s_j$ with probability $p_i \equiv \min\{\lambda x^*_i, 1\}$. 

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Let $x^*$ be an optimal fractional set cover.
Let $\lambda = \ln 2n$.
For $i = 1, 2, \ldots, m$ sequentially do:

*include* or *exclude* $s_i$ — whichever keeps conditional probability of failure below 1.
Given first $t$ choices, probability that elt $e$ won’t be covered is zero if $e$ is already covered, and otherwise

$$\prod_{s_i \ni e, i > t} 1 - p_i.$$ 

Conditional probability that chosen sets will fail to cover is at most

$$\sum_{e \text{ not yet covered}} \prod_{s_i \ni e, i \geq t} 1 - p_i.$$
Given first $t$ choices, expected number of chosen sets is

$$\# \text{ first } t \text{ sets chosen } + \sum_{i > t} p_i.$$  

Given first $t$ choices, probability that too many sets will be chosen is at most

$$\frac{\# \text{ first } t \text{ sets chosen } + \sum_{i > t} p_i}{2 \ln 2n \sum_i x_i^*}.$$  

Let $x^*$ be an optimal fractional set cover.
Let $\lambda = \ln 2n$.
For $i = 1, 2, \ldots, m$ sequentially do:
(include or exclude $s_i$ — whichever keeps
conditional probability of failure below 1.

Given first $t$ choices, probability of failure is at most

$$\Phi_t = \sum_{e \text{ not yet covered}} \prod_{s_i \ni e, i \geq t} (1 - p_i)$$

$$+ \frac{\# \text{ first } t \text{ sets chosen} + \sum_{i > t} p_i}{2 \ln 2n \sum_i x_i^*}.$$
pessimistic estimator $\Phi_t$
Let $x^*$ be an optimal fractional set cover.

Let $\lambda = \ln 2n$.

For $i = 1, 2, \ldots, m$ sequentially do:

*include or exclude $s_i$ — whichever makes $\Phi_i < 1$.*

$$
\Phi_t = \left( \sum_{e \text{ not yet covered}} \prod_{s_i \ni e, i \geq t} 1 - p_i \right) + \frac{\# \text{ first } t \text{ sets chosen} + \sum_{i > t} p_i}{2 \ln 2n \sum_i x_i^*}.
$$

**Corollary**

*Algorithm returns a cover of size at most $2 \ln(2n) \times \text{OPT}$.*
sample and increment
randomized rounding via iterated sampling

Let $x^* \geq 0$ be a fractional solution.
Let $|x^*|$ denote $\sum_i x_i^*$.
Define distribution $p$ by $p_i = x_i^*/\sum_i x_i^*$.
Let $\hat{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots$ do:
  Sample random index $i$ according to $p$.
  Increment $\hat{x}_i$.

Let $\hat{x}^{(t)}$ denote $\hat{x}$ after $t$ samples.

... like weighted balls in bins.
illustration of sampling distribution

fractional set cover $x^*$:

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probability distribution $p$ on sets:

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Let $x^* \geq 0$ be a fractional solution.

Let $|x^*|$ denote $\sum_i x_i^*$.

Define distribution $p$ by $p_i = x_i^* / |x^*|$.

Let $\hat{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots$ do:

Sample random index $i$ according to $p$.
 Increment $\hat{x}_i$ — add $s_i$ to the cover.

Let $\hat{x}^{(t)}$ denote $\hat{x}$ after $t$ samples.
Let $x^* \geq 0$ be a fractional solution.

Let $|x^*|$ denote $\sum_i x_i^*$.

Define distribution $p$ by $p_i \equiv x_i^* / |x^*|$. 

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For $t = 1, 2, 3, \ldots$ do:
  - Sample random index $i$ according to $p$.
  - Increment $\hat{x}_i$ — add $s_i$ to the cover.

Let $\hat{x}^{(t)}$ denote $\hat{x}$ after $t$ samples.

For any element $e$, with each sample,
\[ \Pr[e \text{ is covered}] = \sum_{s_i \ni e} x_i^* / |x^*| \geq 1 / |x^*|. \]
existence proof for set cover

Theorem

With positive probability, after \( T = \lceil \ln(n)|x^*| \rceil \) samples, \( \hat{x}(T) \) is a set cover.

Proof.

For any element \( e \):

- With each sample, \( \Pr[e \text{ is covered}] = \sum_{s_i \ni e} x_i^*/|x^*| \geq 1/|x^*| \).
- After \( T \) samples, \( \Pr[e \text{ is not covered}] \leq (1 - 1/|x^*|)^T < 1/n \).

So, expected number of uncovered elements is less than 1.

Corollary

There exists a set cover of size at most \( \lceil \ln(n)|x^*| \rceil \).
method of conditional probabilities

Let $x^* \geq 0$ be a fractional solution.

Let $\hat{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots, T$ do:

Increment $\hat{x}_i$, where $i$ is chosen to keep expected number of not-covered elements below 1.

Return $\hat{x}(T)$.

Given first $t$ samples, expected number of not-covered elements is at most

$$\Phi_t \doteq \sum_{e \text{ not yet covered}} (1 - 1/|x^*|)^{T-t}.$$
Let $\hat{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots, T$ do:
   Increment $\hat{x}_i$, where $i$ is chosen to minimize
   the number of not-yet-covered elements.
Return $\hat{x}(T)$.

Corollary

The greedy algorithm returns a cover of size at most 
$\lceil \ln(n) \min_{x^*} |x^*| \rceil$. 
Let $\mathbf{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots, T$ do:

- Increment $\hat{x}_i$, where $i$ is chosen to minimize the number of not-yet-covered elements.

Return $\hat{x}(T)$.

**Corollary**

The greedy algorithm returns a cover of size at most $\lceil \ln(n) \min_{x^*} |x^*| \rceil$.

Can also derive Chvatal’s weighted set cover algorithm and show $H(\max_s |s|)$-approximation.
Let $x^*$ be a fractional matching.
Define probability distribution $p$ on edges by $p_e \triangleq \frac{x_e^*}{|x^*|}$.
Let $\hat{x} \leftarrow 0$. Say vertex $v$ is matched when $\sum_{e \ni v} \hat{x}_e = 1$.
Repeat until each edge has a matched vertex:

Sample an edge $e$ from distribution $p$.
If $e$ has no matched vertex, then increment $\hat{x}_e$.
Return $\hat{x}$.
Theorem

The expected size of the matching returned by sample-and-increment is at least $|x^*|/2$.

Proof.

For any edge $e$,

$$\Pr[e \text{ chosen}] = \frac{p_e}{\sum_{e' \cap e' \neq \emptyset} p_{e'}} = \frac{x_e^*}{\sum_{e' \cap e' \neq \emptyset} x_{e'}^*} \geq \frac{x_e^*}{2}.$$
Given the solution $\hat{x}^{(t)}$ after $t$ samples, the expected size $|\hat{x}^{(T)}|$ of the final matching is at least

$$
\Phi_t = |\hat{x}^{(t)}| + \sum_{e \text{ not yet blocked}} x_e^*/2.
$$

Choosing an unblocked edge $(u, v)$ and incrementing $\hat{x}_{(u,v)}$ increases $\Phi$ by at least

$$
1 - \sum_{e \ni u} x_e^*/2 - \sum_{e \ni v} x_e^*/2
\geq 1 - 1/2 - 1/2 = 0.
$$
Let $\hat{x} \leftarrow 0$. Say vertex $v$ is *matched* when $\sum_{e \ni v} \hat{x}_e = 1$.

Repeat until each edge has a matched vertex:

Choose an edge $e$ with no matched vertex. Increment $\hat{x}_e$.

Return $\hat{x}$.

**Corollary**

*The algorithm returns a matching of size at least $(1/2) \max_{x^*} |x^*|$.***
primal: $\max c \cdot x : Ax \leq b$

dual: $\min b \cdot y : A^t y \geq c$

weak duality: $x, y$ feasible $\Rightarrow c \cdot x \leq b \cdot y$, because

\[ c^t x \leq (y^t A)x = y^t (Ax) \leq y^t b. \]

strong duality: Every linear inequality that is valid for all feasible primal solutions $x$ can be expressed via weak duality.
Analysis of algorithm shows $|\hat{x}| \geq |x^*|/2$

for any feasible solution $x^*$.

Analysis must be expressible via weak duality.

weak duality relation for matching $x$ / vertex cover $y$:

$$|x| = \sum_{e} x_e \leq \sum_{e} x_e \sum_{v \in e} y_v = \sum_{v} y_v \sum_{e \ni v} x_e \leq \sum_{v} y_v = |y|.$$  

Find implicit dual solution by looking for coefficients of $x_e^*$ in the inequalities in the analysis.
Dual solution implicit in analysis
look for coefficients of $x^*_e$ in inequalities used in proof

Analysis showed $|x^*|/2 = \Phi_0 \leq \Phi_T = |\hat{x}|$.

Want to recast as weak duality relation for some $\hat{y}$:

$$|x^*| = \sum_e x^*_e \leq \sum_e x^*_e \sum_{v \in e} \hat{y}_v = \sum_v \hat{y}_v \sum_{e \ni v} x^*_e \leq \sum_v \hat{y}_v = |y|.$$ 

Let $e_t = (u_t, v_t)$ be the edge chosen in the $t$th iteration.
Recall $\Phi_T \geq \Phi_0$ proved via $\sum_{t=0}^{T-1} \Phi_{t+1} - \Phi_t \geq 0$, via

$$\sum_t \left(1 - \sum_{e \ni u_t} x^*_e / 2 - \sum_{e \ni v_t} x^*_e / 2 \right) \geq 0.$$ 

Rewrite to isolate coefficients of each $x^*_e$:

$$|\hat{x}| \geq \frac{1}{2} \sum_e x^*_e \sum_{v \in e} \sum_t [v \in e_t].$$ 

Suggests taking $\hat{y}_v = \sum_t [v \in e_t]$, i.e. $\hat{y}_v = 1$ for matched vertices.
implicit primal-dual algorithm

Let \( \hat{x} \leftarrow 0 \). Say vertex \( v \) is matched when \( \sum_{e \ni v} \hat{x}_e = 1 \).

Let \( \hat{y} \leftarrow 0 \).

Repeat until each edge has a matched vertex:

Choose an edge \( e \) with no matched vertex. Increment \( \hat{x}_e \).
For each \( v \in e \), increment \( \hat{y}_v \).

Return \( \hat{x} \).

Corollary

The algorithm returns a feasible vertex cover \( \hat{y} \), with \( |\hat{y}| \leq 2|\hat{x}| \). Thus, the algorithm is a 2-approximation algorithm for vertex cover.
maximum multicommodity flow
input: directed graph $G = (V, E)$, collection $P$ of paths

\[
\text{maximize } \sum_{p \in P} x_p \quad \text{s.t. } (\forall e \in E) \sum_{p \ni e} x_p \leq C
\]

Let $x^*$ be a fractional solution.

Define distribution $q$ on paths by $q_p \equiv x^*_p/|x^*|$. Let $\hat{x} \leftarrow 0$.

For $t = 1, 2, 3, \ldots$ do:

Sample random path $p$ from distribution $q$; increment $\hat{x}_p$. 

existence proof

Theorem

For $T = \lfloor |x^*| \rfloor$ and any $\varepsilon \in [0, 1]$, the expected number of edges on which $\hat{x}(T)$ induces flow greater than $(1 + \varepsilon)C$ is at most

$$m \exp(-\varepsilon^2 C/3).$$

Proof.

Note expected flow on any edge is at most $TC/|x^*| \leq C$. Apply Chernoff.

Corollary

For $\varepsilon = \sqrt{3 \ln(m)/C}$, if $\varepsilon \leq 1$, there exists an integer flow of size at least $\lfloor |x^*| \rfloor$ that induces flow at most $(1 + \varepsilon)C$ on each edge.
Let $\hat{x} \leftarrow 0$. Let $\varepsilon \leftarrow \sqrt{3 \ln(m)/C}$.

Repeat until $\hat{x}$ induces flow of $(1 + \varepsilon)C$ on some edge:

Let $\hat{x}(e)$ denote $\sum_{p \ni e} \hat{x}_p$, the flow on edge $e$.

Choose path $p$ to minimize $\sum_{e \in p} (1 + \varepsilon)\hat{x}(e)$.

Increment $\hat{x}_p$.

Return $\hat{x}$.

**Corollary**

*For $\varepsilon \deq \sqrt{3 \ln(m)/C}$, if $\varepsilon \leq 1$, the algorithm returns an integer flow of size at least $\lfloor \max_{x^*} |x^*| \rfloor$ that induces flow at most $(1 + \varepsilon)C$ on each edge.*
Let $\hat{x} \leftarrow 0$.

Choose $\lambda$ so $\lambda C = 3 \ln(m)/\varepsilon^2$.

Repeat until $\hat{x}$ induces flow of $(1 + \varepsilon)\lambda C$ on some edge:

Let $\hat{x}(e)$ denote $\sum_{p \ni e} \hat{x}_p$, the flow on edge $e$.

Choose path $p$ to minimize $\sum_{e \in p} (1 + \varepsilon)\hat{x}(e)$.

Increment $\hat{x}_p$.

Return $\hat{x}/\lambda$.

**Corollary**

*Given* $\varepsilon \in [0, 1]$, *the algorithm returns a flow of size at least* $\max_{x^*} |x^*|$ *that induces flow at most* $(1 + O(\varepsilon))C$ *on each edge.*

*General alg. requires* $3m \ln(m)/\varepsilon^2$ *shortest-path computations.*
a lower bound on number of iterations

critical dependence on $1/\varepsilon^2$ is inherent?

Define $V(A) \equiv \max\{|x| : Ax \leq 1\}$.

**Theorem**

Let $n \in \mathbb{N}$, $m = n^2$, and $\varepsilon > 0$ such that $\varepsilon^{-2} \leq n^{1-\Omega(1)}$.

Choose $A \in \{0, 1\}^{m \times n}$ uniformly at random.

With probability $1 - o(1)$, for $s \leq \ln(m)/\varepsilon^2$, every $m \times s$ submatrix $B$ of $A$ satisfies

$$V(B) < (1 - \Omega(\varepsilon))V(A).$$

**Proof.**

Discrepancy argument based on “tightness” of Chernoff bound.  $\Box$
a lower bound on number of iterations
\[ \Omega(\log(m)/\varepsilon^2) \] iterations are necessary

Corollary

Let \( n \in \mathbb{N}, m = n^2, \) and \( \varepsilon > 0 \) such that \( \varepsilon^{-2} \leq n^{1-\Omega(1)}. \)

Choose \( A \in \{0, 1\}^{m \times n} \) uniformly at random.

Then with probability \( 1 - o(1), \) for the fractional packing problem of computing \( V(A), \) any \( (1 - \varepsilon) \)-approximate solution \( \hat{x} \) has \( \Omega(\log(m)/\varepsilon^2) \) non-zero entries \( \hat{x}_i. \)
Theorem

A \((1 \pm \varepsilon)\)-approximate primal-dual pair for the linear program \(\max\{c \cdot x : Ax \geq b, x \geq 0\}\) can be computed in expected time

\[
O\left( \#\text{non-zeroes} + n \log(n)/\varepsilon^2 \right)
\]

where \(n = (\#\text{constraints}) + (\#\text{variables})\).

Proof.

Clever use of duality, randomization, algorithmic engineering.

(Strengthens and generalizes result by Grigoriadis and Khachiyan.)
two open questions

- **Set Cover** with *demands and multiplicity constraints* is

\[
\min \{ c \cdot x : Ax \geq b, x \leq 1 \}
\]

where \( A \) is \( \{0, 1\} \).

The greedy algorithm is an \( \ln(n) \)-approximation algorithm.

*Is there a corresponding rounding scheme?*

- For **Facility Location**, the sample-and-increment rounding scheme gives a solution of expected cost at most

\[
\text{assignment-cost}(\text{OPT}) + \ln(n) \times \text{facility-cost}(\text{OPT}).
\]

*Is there a corresponding greedy algorithm?*
set cover

standard randomized rounding
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lower bound on iterations

fast algorithm for explicitly given problems

two open questions